# Spaceability on Morrey Spaces 

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#### Abstract

In this paper, as a main result for Morrey spaces, we prove that the set $\mathcal{M}_{q}^{p}\left(\mathbb{R}^{n}\right) \backslash \bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}\left(\mathbb{R}^{n}\right)$ is spaceable in $\mathcal{M}_{q}^{p}\left(\mathbb{R}^{n}\right)$, where $0<q<p<\infty$.


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## 1. Introduction

Let $0<q \leq p<\infty$. Define the Morrey (quasi-) norm $\|\cdot\|_{\mathcal{M}_{q}^{p}}$ by

$$
\|f\|_{\mathcal{M}_{q}^{p}} \equiv \sup \left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^{q}(Q)}: \quad Q \text { is a cube in } \mathbb{R}^{n}\right\}
$$

for a measurable function $f$, where a cube is defined to be the set of the form $\left\{a+y: a \in \mathbb{R}^{n}, y \in[0, t]^{n}\right\}$ for some $a \in \mathbb{R}^{n}$ and $t>0$. The Morrey space $\mathcal{M}_{q}^{p}$ is the set of all measurable functions $f$ for which $\|f\|_{\mathcal{M}_{q}^{p}}$ is finite. Morrey spaces date back to 1938 , when C.B. Morrey considered elliptic differential equations and discussed continuity of the solutions using a lemma [7]. His lemma was refined by Peetre [8]. Morrey's lemma gave rise to the theory of function spaces;

[^0]see [8] and [10]. The function spaces dealt with are called Morrey spaces. Let $0<q<p<\infty$. For each $r \in(q, p]$ the set $\mathcal{M}_{r}^{p}$ is a proper subset of $\mathcal{M}_{q}^{p}[4,9]$. In this paper, we show that the difference of these two sets is large enough. Precisely, by a techmical lemma we prove:

Theorem 1.1. If $0<q<p$, then $\mathcal{M}_{q}^{p} \backslash \bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}$ is a spaceable subset of $\mathcal{M}_{q}^{p}$.

Another different proof will also be given via [6, Theorem 3.3]. We recall that a subset $S$ of a topological vector space $X$ is called spaceable if $S \cup\{0\}$ contains a closed infinite-dimensional linear subspace of $X$. The concepts spaceability and lineability were introduced by the paper [1] and then have been studied on different kinds of function or sequence spaces (see $[5,6]$ ). In particular, for spaceability of the difference of Lebesgue spaces (see [2, 3, 11]).

## 2. Main Result

Let $p>q>0$ and $R>1$ be fixed so that

$$
\begin{equation*}
(R+1)^{-\frac{1}{p}}=2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}} . \tag{2.1}
\end{equation*}
$$

For a vector $\varepsilon \in\{0,1\}^{n}$, we define an affine transformation $T_{\varepsilon}$ by

$$
T_{\varepsilon}(x) \equiv \frac{1}{R+1} x+\frac{R}{R+1} \varepsilon \quad\left(x \in \mathbb{R}^{n}\right)
$$

Let $E_{0}:=[0,1]^{n}$. Suppose that we have defined $E_{0}, E_{1}, E_{2}, \ldots, E_{j}$. Define

$$
E_{j+1}:=\bigcup_{\varepsilon \in\{0,1\}^{n}} T_{\varepsilon}\left(E_{j}\right)
$$

and

$$
E_{j, 0}:=\left[0,(1+R)^{-j}\right]^{n}
$$

The following technical lemma is proved in [10, Proposition 4.1]. Here, for the sake of convenience of readers, we reproduce the proof.

Lemma 2.1. Under above notations we have

$$
\begin{equation*}
\left\|\chi_{E_{j}}\right\|_{\mathcal{M}_{q}^{p}} \sim(1+R)^{-j \frac{n}{p}}=\left\|\chi_{E_{j, 0}, 0}\right\|_{\mathcal{M}_{q}^{p}}=\left\|\chi_{E_{j, 0}}\right\|_{p}=\left\|\chi_{E_{j}}\right\|_{q}, \tag{2.2}
\end{equation*}
$$

where the implicit constants in $\sim$ does not depend on $j$ but can depend on $p$ and $q$ and $\|\cdot\|_{\mathcal{M}_{q}^{p}}$ is the Morrey norm.

Proof. A direct calculation shows that

$$
\left\|\chi_{E_{j}}\right\|_{\mathcal{M}_{q}^{p}} \geq\left\|\chi_{E_{j, 0}}\right\|_{\mathcal{M}_{q}^{p}}=(1+R)^{-j \frac{n}{p}}=\left\|\chi_{E_{j, 0}}\right\|_{p}=\left\|\chi_{E_{j}}\right\|_{q},
$$

Thus, we need to show

$$
\left\|\chi_{E_{j}}\right\|_{\mathcal{M}_{q}^{p}} \lesssim\left\|\chi_{E_{j, 0}}\right\|_{p} .
$$

Let us calculate

$$
\left\|\chi_{E_{j}}\right\|_{\mathcal{M}_{q}^{p}} \sim \sup _{S \in \mathcal{Q}}|S|^{\frac{1}{p}-\frac{1}{q}}\left|S \cap E_{j}\right|^{\frac{1}{q}},
$$

where $\mathcal{Q}$ denotes the set of all cubes. Fix $j \in \mathbb{N}$. Let us temporally say that $Q \in \mathcal{Q}$ is wasteful, if there exists a cube $S \in \mathcal{Q}$ such that

$$
|Q|^{\frac{1}{p}-\frac{1}{q}}\left|Q \cap E_{j}\right|^{\frac{1}{q}}<|S|^{\frac{1}{p}-\frac{1}{q}}\left|S \cap E_{j}\right|^{\frac{1}{q}}
$$

Thus by definition, if $\ell(Q):=|Q|^{\frac{1}{n}}>1$, then

$$
|Q|^{\frac{1}{p}-\frac{1}{q}}\left|Q \cap E_{j}\right|^{\frac{1}{q}}<\left|E_{j}\right|^{\frac{1}{q}}=\left|[0,1]^{n}\right|^{\frac{1}{p}-\frac{1}{q}}\left|[0,1]^{n} \cap E_{j}\right|^{\frac{1}{q}}
$$

In addition, if the side-length of a cube $Q$ is less than $(R+1)^{-j}$, then it is wasteful. Indeed, then the equalities

$$
\begin{aligned}
& \sup \left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\left|Q \cap E_{j}\right|^{\frac{1}{q}}: Q \in \mathcal{Q},|Q| \leq(R+1)^{-j n}\right\} \\
& =\sup \left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\left|Q \cap E_{j}\right|^{\frac{1}{q}}: Q \in \mathcal{Q}, Q \subset E_{j}\right\} \\
& =\sup \left\{|Q|^{\frac{1}{p}}: Q \in \mathcal{Q}, Q \subset E_{j}\right\} \\
& =\left|E_{j, 0}\right|^{\frac{1}{p}}
\end{aligned}
$$

hold. Here, we obtain the first equality by translating $Q$ so that $Q$ which is included in $E_{j, 0}$. This calculation shows that

$$
|Q|^{\frac{1}{p}-\frac{1}{q}}\left|Q \cap E_{j}\right|^{\frac{1}{q}}<\left|E_{j, 0}\right|^{\frac{1}{p}}
$$

for any such cube. Thus, if the cube $Q$ is not wasteful, then $(R+1)^{-j} \leq$ $\ell(Q) \leq 1$. So, there exists $k \in\{1,2, \ldots, n\}$ such that $(R+1)^{-k n} \leq|Q| \leq$ $(R+1)^{-(k-1) n}$. In this case, since any connected component $P$ of $E_{k}$ satisfies $(R+1)^{-k n}=|P| \leq|Q| \leq(R+1)^{-(k-1) n}, 3 Q$ contains a connected component of $E_{k}$. Hence, it follows that

$$
\begin{aligned}
& \left\|\chi_{E_{j}}\right\|_{\mathcal{M}_{q}^{p}} \\
& \sim \sup \left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\left|Q \cap E_{j}\right|^{\frac{1}{q}}: Q \text { contains a connected component of } E_{j}\right\} .
\end{aligned}
$$

Let $S$ be a cube which contains a connected component of $E_{j}$ and is not wasteful. By symmetry, we may assume $S=I \times I \times \cdots \times I$ for some interval $I$. We define

$$
S^{*}:=\operatorname{co}\left(\bigcup\left\{W: W \text { is a connected component of } E_{j} \text { intersecting } S\right\}\right)
$$

where $\operatorname{co}(A)$ stands for the smallest convex set containing a set $A$. Then a geometric observation shows that $S^{*}$ engulfs $k^{n}$ connected component of $E_{j}$ for some $1 \leq k \leq 2^{j}$. Take an integer $l$ such that $2^{l-1} \leq k \leq 2^{l}$. Then we have

$$
\left|S^{*} \cap E_{j}\right|=k^{n}(1+R)^{-j n}, \quad\left|S^{*}\right| \sim(1+R)^{-j n+l n}
$$

Consequently, from (2.1) we have

$$
\begin{aligned}
\left|S^{*}\right|^{\frac{1}{p}-\frac{1}{q}}\left|S^{*} \cap E_{j}\right|^{\frac{1}{q}} & \sim 2^{\frac{l n}{q}}(1+R)^{-\frac{j n}{q}}(1+R)^{(-j+l)\left(\frac{n}{p}-\frac{n}{q}\right)} \\
& =2^{\frac{l n}{q}}(1+R)^{-j \frac{n}{p}+\ln \left(\frac{1}{p}-\frac{1}{q}\right)} \\
& =2^{\frac{l n}{q}}(1+R)^{\ln \left(\frac{1}{p}-\frac{1}{q}\right)}(1+R)^{-j \frac{n}{p}}=(1+R)^{-j \frac{n}{p}} .
\end{aligned}
$$

Therefore, (2.2) is obtained.
So, we can say that the Morrey norm $\|\cdot\|_{\mathcal{M}_{q}^{p}}$ reflects local regularity of the functions more precisely than the Lebesgue norm $\|\cdot\|_{p}$.

A chain of equalities in (2.2) is the motivation of choosing $R$ in (2.1).
Remark 2.2. In Lemma 2.1, if one defines $F_{j}:=\left\{x \in \mathbb{R}^{n}:(R+1)^{-j} x \in E_{j}\right\}$, then $\left\{F_{j}\right\}_{j=1}^{\infty}$ is an increasing sequence of sets and each $F_{j}$ is made up of disjoint union of cubes of length $1,\left\|\chi_{F_{j}}\right\|_{\mathcal{M}_{q}^{p}} \sim 1$, and each component of $F_{j}$ is a cube of size 1 .

Proposition 2.3. Let $0<q<r \leq p$. Then $\left\|\chi_{F_{j}}\right\|_{\mathcal{M}_{r}^{p}} \geq\left(\frac{1+R}{2}\right)^{\left(\frac{n}{q}-\frac{n}{r}\right) j}$, and so that $\lim _{j \rightarrow \infty}\left\|\chi_{F_{j}}\right\|_{\mathcal{M}_{r}^{p}}=\infty$.

Proof. Simply use

$$
\left\|\chi_{F_{j}}\right\|_{\mathcal{M}_{r}^{p}} \geq(1+R)^{\frac{j n}{p}-\frac{j n}{r}}\left|F_{j}\right|^{\frac{1}{r}}=(1+R)^{\frac{j n}{q}-\frac{j n}{r}}\left|F_{j}\right|^{\frac{1}{r}-\frac{1}{q}}=\left(\frac{1+R}{2}\right)^{\left(\frac{n}{q}-\frac{n}{r}\right) j}
$$

Now, we prove the main result of this paper.
Proof of Theorem 1.1. Under above notations, put

$$
\begin{equation*}
F:=\bigcup_{j=1}^{\infty} F_{j} . \tag{2.3}
\end{equation*}
$$

Then $\chi_{F} \in \mathcal{M}_{q}^{p}$. For each $j=1,2,3, \ldots$ let us define

$$
g_{j}(x):=\chi_{F}\left((R+1)^{-j-1} x\right)-\chi_{F}\left((R+1)^{-j} x\right) \quad\left(x \in \mathbb{R}^{n}\right)
$$

and

$$
h_{j}(x):=\chi_{F}\left(R^{-j-1} x\right) \quad\left(x \in \mathbb{R}^{n}\right)
$$

Then by Lemma 2.1 we have $\left\|g_{j}\right\|_{\mathcal{M}_{q}^{p}} \sim\left\|h_{j}\right\|_{\mathcal{M}_{q}^{p}}<\infty$, while $\left\|g_{j}\right\|_{\mathcal{M}_{r}^{p}} \sim$ $\left\|h_{j}\right\|_{\mathcal{M}_{r}^{p}}=\infty$ for all $q<r \leq p$. We set

$$
V:=\left\{\sum_{j=1}^{\infty} \lambda_{j} g_{j}: \text { for all } j, \lambda_{j} \in \mathbb{C}\right\} \bigcap \mathcal{M}_{q}^{p}
$$

Since the convergence with norm of $\mathcal{M}_{q}^{p}$ is stronger than the almost eveywhere convergence, $V$ is closed in $\mathcal{M}_{q}^{p}$. Also, note that

$$
V \cap \bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}=\{0\},
$$

since the charactristic function $\chi_{F}$ belongs to the set $\mathcal{M}_{q}^{p} \backslash \bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}$. Therefore, $\mathcal{M}_{q}^{p} \backslash \bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}$ is spaceable in $\mathcal{M}_{q}^{p}$.

Here is an alternative proof of Thoerem 1.1 with $q \geq 1$ using a result by Kitson and Timoney [6]. We invoke this result from [6].

Theorem 2.4. [6, Theorem 3.3]. Let $Z_{m}(m \in \mathbb{N})$ be Banach spaces and $X$ a Fréchet space. Let $T_{m}: Z_{m} \rightarrow X$ be continuous linear operators and $Y$ the linear span of $\bigcup_{m=1}^{\infty} T_{m}\left(Z_{m}\right)$. If $Y$ is not closed in $X$, then the complement $X \backslash Y$ is spaceable.

For a different proof of Theorem 1.1, simply apply Theorem 2.4 with

$$
X=\mathcal{M}_{q}^{p}, \quad Z_{m}=\mathcal{M}_{q+\frac{p-q}{m}}^{p}, \quad T_{m}=\text { inclusion mapping from } Z_{m} \text { to } \mathcal{M}_{q}^{p}
$$

We will check the following property of the Morrey space $\mathcal{M}_{q}^{p}$ to see that Theorem 2.4 is applicable.

Proposition 2.5. Let $0<q<p<\infty$. Then $\bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}$ is not closed in $\mathcal{M}_{q}^{p}$.
We torelate the case $0<q<1$ here. Once this is shown, Theorem 1.1 will be proved with the help of the aforementioned theorem.

Proof. Let $F_{j}$ be as in Remark 2.2. For each $k \in \mathbb{N}$ define

$$
f_{k}:=\sum_{j=1}^{k} \frac{1}{j^{2}\left\|\chi_{F_{j}}\right\|_{\mathcal{M}_{q}^{p}}} \chi_{F_{j}} .
$$

Then $\left(f_{k}\right)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_{q}^{p}$. So, $\left(f_{k}\right)_{k=1}^{\infty}$ converges to a function $f$ in $\mathcal{M}_{q}^{p}$ since $\mathcal{M}_{q}^{p}$ is a Banach space. Note that each $f_{k} \in L^{p} \subset \mathcal{M}_{r}^{p}$ for all $q<r \leq p$. If $f$ is a member in $\mathcal{M}_{r}^{p}$ for some $r \in(q, p]$, then

$$
\left\{\frac{1}{j^{2}\left\|\chi_{F_{j}}\right\|_{\mathcal{M}_{q}^{p}}} \chi_{F_{j}}\right\}_{j=1}^{\infty}
$$

would form a bounded set in $\mathcal{M}_{r}^{p}$ since $\mathcal{M}_{r}^{p}$ enjoys the lattice property. This is a contradiction to Proposition 2.3.

## 3. Appendix

The situation is different from the case of Lebesgue spaces.
Proposition 3.1. Let $0<q<p<\infty$. Then $\bigcup_{q<r \leq p} \mathcal{M}_{r}^{p}$ is not dense in $\mathcal{M}_{q}^{p}$.
Proof. Let $F$ be as in (2.3). We prove that $2 \chi_{F}$ is not in the closure of $\bigcup \mathcal{M}_{r}^{p}$ $q<r \leq p$
by showing that for every $q<r \leq p, f \notin \mathcal{M}_{r}^{p}$ if $f \in \mathcal{M}_{q}^{p}$ satisfies $\left\|2 \chi_{F}-f\right\|_{\mathcal{M}_{q}^{p}}<$ 1. Indeed, if $K$ is one of the connected components of $F$, then $\|f\|_{L^{r}(K)} \geq$ $\|f\|_{L^{q}(K)}>c_{q}=\left(2^{\min (1, q)}-1\right)^{\frac{1}{\min (1, q)}}$ since

$$
\begin{aligned}
1 & >\left(\left\|2 \chi_{F}-f\right\|_{\mathcal{M}_{q}^{p}}\right)^{\min (1, q)} \\
& \geq\left(\|2-f\|_{L^{q}(K)}\right)^{\min (1, q)} \\
& \geq 2^{\min (1, q)}-\left(\|f\|_{L^{q}(K)}\right)^{\min (1, q)}
\end{aligned}
$$

Thus,

$$
\|f\|_{\mathcal{M}_{r}^{p}} \geq\left|\left[0, R^{j}\right]^{n}\right|^{\frac{1}{p}-\frac{1}{r}}\|f\|_{L^{r}\left(\left[0, R^{j}\right]^{n}\right)} \geq c_{q} R^{\frac{j n}{p}-\frac{j n}{r}}\left|E_{j}\right|^{\frac{1}{r}}=c_{q} 2^{\frac{j n}{r}-\frac{j n}{q}} R^{\frac{j n}{q}-\frac{j n}{r}}
$$

for all $j \in \mathbb{N}$. Hence, $\|f\|_{\mathcal{M}_{r}^{p}}=\infty$, or equivalently $f \notin \mathcal{M}_{r}^{p}$.

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