

## $n$ -submodules

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ABSTRACT. Let  $R$  be a commutative ring with identity. A proper submodule  $N$  of an  $R$ -module  $M$  is an  $n$ -submodule if  $rm \in N$  ( $r \in R, m \in M$ ) with  $r \notin \sqrt{\text{Ann}_R(M)}$ , then  $m \in N$ . A number of results concerning  $n$ -submodules are given. For example, we give other characterizations of  $n$ -submodules. Also various properties of  $n$ -submodules are considered.

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### 1. INTRODUCTION

Throughout this article,  $R$  denotes a commutative ring with identity and all modules are unitary. Also  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$  will denote, respectively, the natural numbers, the ring of integers, and the field of rational numbers. If  $N$  is an  $R$ -submodule of  $M$ , annihilator of  $R$ -module  $\frac{M}{N}$  is defined to be  $\text{Ann}_R(\frac{M}{N}) = (N :_R M) = \{r \in R : rM \subseteq N\}$ . Also the annihilator of  $M$ , denoted by  $\text{Ann}_R(M)$ , is  $(0 :_R M)$ . Suppose that  $I$  is an ideal of  $R$ . We denote the radical of  $I$  by  $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$ .

A proper submodule  $N$  of  $M$  is called prime (*primary*) if  $rx \in N$ , for  $r \in R$  and  $x \in M$ , implies that either  $x \in N$  or  $r \in (N :_R M)$  ( $r^n \in (N :_R M)$ ), for

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some  $n \in \mathbb{N}$  (see [1], [6], [9], [11]).

An  $R$ -module  $M$  is said to be a multiplication module, if for each submodule  $N$  of  $M$ , there is an ideal  $I$  of  $R$ , such that  $N = IM$ . Equivalently,  $M$  is a multiplication module if and only if  $N = (N :_R M)M$ , for each submodule  $N$  of  $M$  [2],[3].

The concepts of  $n$ -ideals and  $n$ -submodules were introduced in [12]. A proper ideal  $I$  of  $R$  is said to be an  $n$ -ideal if the condition  $ab \in I$  with  $a \notin \sqrt{0} = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$  implies  $b \in I$ , for every  $a, b \in R$ . Also a proper submodule  $N$  of  $M$  is called an  $n$ -submodule if for  $a \in R$ ,  $x \in M$ ,  $ax \in N$  with  $a \notin \sqrt{\text{Ann}_R(M)}$ , then  $x \in N$ .

In Section 2, we investigate some properties of  $n$ -submodules analogous with  $n$ -ideals and also obtain some basic results. Among many results in this article, it is shown in Theorem 2.2, that a proper submodule  $N$  of  $M$  is an  $n$ -submodule if and only if  $N = (N :_M a)$  for every  $a \notin \sqrt{\text{Ann}_R(M)}$ . In Theorem 2.22, we show that every  $n$ -submodule is a primary submodule. Furthermore, in Theorem 2.27, we characterize torsion-free modules in terms of  $n$ -submodules.

## 2. $n$ -SUBMODULES

Recall that a proper submodule  $N$  of a module  $M$  over a commutative ring  $R$  is said to be an  $n$ -submodule, if for  $a \in R$ ,  $x \in M$ ,  $ax \in N$  with  $a \notin \sqrt{\text{Ann}_R(M)}$ , then  $x \in N$ .

EXAMPLE 2.1. (i) Suppose that  $R$  is a ring that has only one prime ideal. Then every proper submodule of  $R$ -module  $R$  is an  $n$ -submodule.

(ii)  $\mathbb{Z}_6$  as  $\mathbb{Z}$ -module has not any  $n$ -submodule.

**Theorem 2.2.** *Let  $M$  be an  $R$ -module and  $N$  be a proper submodule of  $M$ . Then the following statements are equivalent:*

- (i)  $N$  is an  $n$ -submodule of  $M$ ;
- (ii)  $N = (N :_M a)$ , for every  $a \notin \sqrt{\text{Ann}_R(M)}$ ;
- (iii) For any ideal  $I$  of  $R$  and submodule  $K$  of  $M$ ,  $IK \subseteq N$  with  $I \not\subseteq \sqrt{\text{Ann}_R(M)}$  implies  $K \subseteq N$ .

*Proof.* (i)  $\Rightarrow$  (ii) Let  $N$  be an  $n$ -submodule of  $M$ . For every  $a \in R$ , the inclusion  $N \subseteq (N :_M a)$  always holds. Let  $a \notin \sqrt{\text{Ann}_R(M)}$  and  $x \in (N :_M a)$ . Then we have  $ax \in N$ . Since  $N$  is an  $n$ -submodule, we conclude that  $x \in N$  and thus  $N = (N :_M a)$ .

(ii)  $\Rightarrow$  (iii) Suppose that  $IK \subseteq N$  where  $I \not\subseteq \sqrt{\text{Ann}_R(M)}$ , for ideal  $I$  of  $R$  and submodule  $K$  of  $M$ . Since  $I \not\subseteq \sqrt{\text{Ann}_R(M)}$ , there exists  $a \in I$  such that  $a \notin \sqrt{\text{Ann}_R(M)}$ . Then we have  $aK \subseteq N$ , and so  $K \subseteq (N :_M a) = N$  by (ii).

(iii)  $\Rightarrow$  (i) Let  $ax \in N$  with  $a \notin \sqrt{\text{Ann}_R(M)}$  for  $a \in R$  and  $x \in M$ . It is sufficient to take  $I := Ra$  and  $K := Rx$  to prove the result.  $\square$

**Proposition 2.3.** *i) If  $N$  is an n-submodule of  $M$ , then  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$ .  
 ii) Let  $\{N_i\}_{i \in I}$  be a nonempty set of n-submodules of an  $R$ -module  $M$ . Then  $\bigcap_{i \in I} N_i$  is an n-submodule.  
 iii) Let  $\{N_i\}_{i \in I}$  be a chain of n-submodules of a finitely generated  $R$ -module  $M$ . Then  $\bigcup_{i \in I} N_i$  is an n-submodule of  $M$ .*

*Proof.* i) Assume that  $N$  is an n-submodule; but  $(N :_R M) \not\subseteq \sqrt{\text{Ann}_R(M)}$ . Then there exists  $r \in (N :_R M)$  such that  $r \notin \sqrt{\text{Ann}_R(M)}$ . Thus  $rM \subseteq N$  and since  $N$  is an n-submodule, we conclude that  $N = M$ , a contradiction. Hence  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$ .

ii) Let  $rx \in \bigcap_{i \in I} N_i$  with  $r \notin \sqrt{\text{Ann}_R(M)}$ , for  $r \in R$  and  $x \in M$ . Then  $rx \in N_i$ , for every  $i \in I$ . Since for every  $i \in I$ ,  $N_i$  is an n-submodule of  $M$ , we get  $x \in N_i$  and so  $x \in \bigcap_{i \in I} N_i$ .

iii) Let  $rx \in \bigcup_{i \in I} N_i$  where  $r \notin \sqrt{\text{Ann}_R(M)}$  for  $r \in R$  and  $x \in M$ . Then  $rx \in N_k$  for some  $k \in \mathbb{N}$ . Since  $N_k$  is an n-submodule, we conclude that  $x \in N_k \subseteq \bigcup_{i \in I} N_i$  and so  $\bigcup_{i \in I} N_i$  is an n-submodule.  $\square$

**Proposition 2.4.** *Let  $I$  be an ideal of  $R$  such that  $I \not\subseteq \sqrt{\text{Ann}_R(M)}$ . Then the followings hold:*

- (i) *If  $K_1$  and  $K_2$  are n-submodules of  $M$  with  $IK_1 = IK_2$ , then  $K_1 = K_2$ .*
- (ii) *If  $IK$  is an n-submodule of  $M$ , then  $IK = K$ .*

*Proof.* (i) Since  $K_1$  is an n-submodule and  $IK_2 \subseteq K_1$ , by Theorem 2.2, we get that  $K_2 \subseteq K_1$ . Likewise,  $K_1 \subseteq K_2$ .

(ii) Since  $IK$  is an n-submodule and  $IK \subseteq IK$ , we conclude that  $K \subseteq IK$ , so this completes the proof.  $\square$

The next lemma provides a useful characterization of modules that have n-submodule.

**Lemma 2.5.** *Let  $M$  be a torsion-free  $R$ -module. Then zero submodule is an n-submodule of  $M$ .*

*Proof.* Let  $ax = 0$  with  $a \notin \sqrt{\text{Ann}_R(M)}$ , for  $a \in R$  and  $x \in M$ . Since  $M$  is torsion-free,  $x = 0$ . Thus zero submodule of  $M$  is an n-submodule.  $\square$

**Lemma 2.6.** *If  $M$  is a torsion-free multiplication  $R$ -module, then zero submodule is the only n-submodule of  $M$ .*

*Proof.* Suppose that  $N$  is an n-submodule of  $M$ . Then by Proposition 2.3(i), we have  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)} = 0$  and so  $(N :_R M) = 0$ . As  $M$  is multiplication, then  $N = 0$ . So by Lemma 2.5, the zero submodule is the only n-submodule.  $\square$

**Proposition 2.7.** *Let  $M$  be an  $R$ -module and  $I$  be an ideal of  $R$ . If  $N$  is an  $n$ -submodule of  $M$  such that  $I \not\subseteq (N :_R M)$ , then  $(N :_M I)$  is an  $n$ -submodule of  $M$ .*

*Proof.* Let  $ax \in (N :_M I)$  with  $a \notin \sqrt{\text{Ann}_R(M)}$ , for  $a \in R$  and  $x \in M$ . So  $aIx \subseteq N$  and as  $N$  is an  $n$ -submodule,  $Ix \subseteq N$ . Hence  $x \in (N :_M I)$ .  $\square$

**Proposition 2.8.** *Let  $N$  be a proper submodule of  $M$ . Then  $N$  is an  $n$ -submodule if and only if for every  $x \in M$ ,  $(N :_R x) = R$  or  $(N :_R x) \subseteq \sqrt{\text{Ann}_R(M)}$ .*

*Proof.* Assume that  $N$  is an  $n$ -submodule. If  $(N :_R x) \not\subseteq \sqrt{\text{Ann}_R(M)}$ , then there exists  $r \in (N :_R x) - \sqrt{\text{Ann}_R(M)}$ . So  $rx \in N$  where  $r \notin \sqrt{\text{Ann}_R(M)}$ . Since  $N$  is an  $n$ -submodule,  $x \in N$ . Hence  $(N :_R x) = R$ . Conversely, let  $rx \in N$  where  $r \notin \sqrt{\text{Ann}_R(M)}$ , for  $r \in R$  and  $x \in M$ . So  $r \in (N :_R x) - \sqrt{\text{Ann}_R(M)}$ . By assumption, we have  $(N :_R x) = R$  and therefore  $x \in N$ .  $\square$

**Corollary 2.9.** *Let  $N$  be a proper submodule of  $M$ . Then  $N$  is an  $n$ -submodule if and only if for every  $x \in M - N$ ,  $(N :_R x) \subseteq \sqrt{\text{Ann}_R(M)}$ .*

Recall that,  $r \in R$  is said to be a zero divisor of an  $R$ -module  $M$ , if there exists a non-zero element  $x \in M$  such that  $rx = 0$ .

**Theorem 2.10.** *Let  $M$  be an  $R$ -module and  $N$  be a submodule of  $M$ . Then  $N$  is an  $n$ -submodule if and only if every zero divisor of an  $R$ -module  $\frac{M}{N}$  is in  $\sqrt{\text{Ann}_R(M)}$ .*

*Proof.* Let  $N$  be an  $n$ -submodule and  $r$  be a zero divisor of  $\frac{M}{N}$ . Then there exists  $x \in M - N$  such that  $rx \in N$ . Since  $N$  is an  $n$ -submodule, we have  $r \in \sqrt{\text{Ann}_R(M)}$ . For the converse, assume that  $rx \in N$  where  $x \notin N$ , for  $r \in R$  and  $x \in M$ . Then  $r$  is a zero divisor of  $\frac{M}{N}$  and so  $r \in \sqrt{\text{Ann}_R(M)}$ .  $\square$

**Theorem 2.11.** *Every maximal  $n$ -submodule is a prime submodule.*

*Proof.* Let  $N$  be a maximal  $n$ -submodule of  $M$  and  $ax \in N$  where  $a \notin (N :_R M)$ , for  $a \in R$  and  $x \in M$ . By Proposition 2.7,  $(N :_M a)$  is an  $n$ -submodule. Thus  $x \in (N :_M a) = N$ , by maximality of  $N$ . So  $N$  is a prime submodule.  $\square$

**Theorem 2.12.** *Let  $M$  be a finitely generated  $R$ -module. If  $M$  has an  $n$ -submodule, then  $M$  has a prime submodule.*

*Proof.* Suppose that  $N$  is an  $n$ -submodule and  $\Omega = \{L : L \text{ is an } n\text{-submodule of } M; N \subseteq L\}$ . By Zorn's Lemma,  $\Omega$  has a maximal element  $K \in \Omega$ . Then by Theorem 2.11,  $K$  is a prime submodule of  $M$ .  $\square$

In ring theory (and so in module theory), the concepts prime ideal and  $n$ -ideal are not the same in general. (see Example 3.2 in [12]). In the following, we try to find some relations between them.

**Proposition 2.13.** *For a prime submodule  $N$  of  $M$ ,  $N$  is an  $n$ -submodule if and only if  $(N :_R M) = \sqrt{\text{Ann}_R(M)}$ .*

*Proof.* Suppose that  $N$  is a prime submodule of  $M$ . It is clear that  $\sqrt{\text{Ann}_R(M)} \subseteq (N :_R M)$ . If  $N$  is an  $n$ -submodule, then by Proposition 2.3(i), we have  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$  and so  $(N :_R M) = \sqrt{\text{Ann}_R(M)}$ . For the converse, assume that  $(N :_R M) = \sqrt{\text{Ann}_R(M)}$ . Now we show that  $N$  is an  $n$ -submodule. Let  $ax \in N$  and  $a \notin \sqrt{\text{Ann}_R(M)}$ , for  $a \in R$  and  $x \in M$ . Since  $N$  is a prime submodule and  $a \notin (N :_R M)$ , we get  $x \in N$  and so  $N$  is an  $n$ -submodule.  $\square$

Recall from [11], the intersection of all prime submodules contains  $N$ , denoted  $\text{rad}(N)$ , is called the radical of  $N$ . If there is no prime submodule containing  $N$ ,  $\text{rad}(N) = M$ .

**Proposition 2.14.** *Let  $M$  be a finitely generated  $R$ -module. Then  $\text{rad}(0)$  is an  $n$ -submodule if and only if  $\text{rad}(0)$  is a prime submodule.*

*Proof.* Since  $M$  is finitely generated, by Theorem 4.4 in [8],  $(\text{rad}(0) :_R M) = \sqrt{\text{Ann}_R(M)}$ . Suppose that  $\text{rad}(0)$  is an  $n$ -submodule. Let  $ax \in \text{rad}(0)$  with  $a \notin (\text{rad}(0) :_R M)$ , for  $a \in R$  and  $x \in M$ . So  $a \notin \sqrt{\text{Ann}_R(M)}$  and since  $\text{rad}(0)$  is an  $n$ -submodule, we have  $x \in \text{rad}(0)$ . Thus  $\text{rad}(0)$  is a prime submodule. Now assume that  $\text{rad}(0)$  is a prime submodule. By Proposition 2.13,  $\text{rad}(0)$  is an  $n$ -submodule.  $\square$

**Lemma 2.15.** *Let  $N$  be an  $n$ -submodule of an  $R$ -module  $M$  such that  $(N :_R M) = \sqrt{\text{Ann}_R(M)}$ . Then  $N$  is a prime submodule.*

*Proof.* It is clear.  $\square$

**Proposition 2.16.** *If zero submodule of an  $R$ -module  $M$  is an  $n$ -submodule, then  $\sqrt{\text{Ann}_R(M)}$  is a prime ideal of  $R$ .*

*Proof.* Let  $ab \in \sqrt{\text{Ann}_R(M)}$  for  $a, b \in R$ . So there exists  $n \in \mathbb{N}$  such that  $a^n b^n M = 0$ . If  $a \notin \sqrt{\text{Ann}_R(M)}$ , then since the zero submodule is a  $n$ -submodule, we get  $b^n M = 0$ ; i.e.  $b \in \sqrt{\text{Ann}_R(M)}$ .  $\square$

Remember that if  $N$  is a prime submodule of an  $R$ -module  $M$ , then  $(N :_R M)$  is a prime ideal of  $R$ . Now, we give a similar result for  $n$ -submodules.

**Lemma 2.17.** *If  $M$  is a faithful  $R$ -module and  $N$  is an  $n$ -submodule of  $M$ , then  $(N :_R M)$  is an  $n$ -ideal of  $R$ .*

*Proof.* Assume that  $ab \in (N :_R M)$  with  $a \notin \sqrt{0}$ , for  $a, b \in R$ . Since  $\text{Ann}_R(M) = 0$  and  $N$  is an  $n$ -submodule, then  $b \in (N :_R M)$ .  $\square$

**Corollary 2.18.** *Let  $M$  be a faithful  $R$ -module and  $R$  has no  $n$ -ideal. Then  $M$  has no  $n$ -submodule.*

**Lemma 2.19.** *Let  $M$  be a multiplication  $R$ -module and  $N$  be a submodule of  $M$  such that  $(N :_R M)$  is an  $n$ -ideal of  $R$ . Then  $N$  is an  $n$ -submodule.*

*Proof.* Let  $IK \subseteq N$  with  $I \not\subseteq \sqrt{\text{Ann}_R(M)}$ , where  $I$  is an ideal of  $R$  and  $K$  is a submodule of  $M$ . Since  $M$  is multiplication and  $(N :_R M)$  is an  $n$ -ideal,  $I(K :_R M) \subseteq (N :_R M)$  and so  $(K :_R M) \subseteq (N :_R M)$ , by Theorem 2.7 in [12]. Thus  $K \subseteq N$  and by Theorem 2.2,  $N$  is an  $n$ -submodule.  $\square$

**Corollary 2.20.** *Let  $M$  be a cyclic  $R$ -module and  $N$  be a submodule of  $M$  such that  $(N :_R M)$  is an  $n$ -ideal of  $R$ . Then  $N$  is an  $n$ -submodule of  $M$ .*

Recall that a proper submodule  $N$  of  $M$  is said to be an  $r$ -submodule, if for  $a \in R$ ,  $m \in M$  and whenever  $am \in N$  with  $\text{ann}_M(a) = 0$ , then  $m \in N$  [5].

**Proposition 2.21.** *Every  $n$ -submodule is an  $r$ -submodule.*

*Proof.* Let  $N$  be an  $n$ -submodule of  $M$ . Now, we will show that  $N$  is an  $r$ -submodule. Let  $am \in N$  with  $\text{ann}_M(a) = 0$ , for some  $a \in R$ ,  $m \in M$ . Assume that  $a \in \sqrt{\text{Ann}_R(M)}$ . Then there exists  $n \in \mathbb{N}$  such that  $a^n M = 0$ . Choose the smallest positive integer  $n$  such that  $a^n M = 0$ . Then we have  $a^{n-1} M \neq 0$ . Since  $a(a^{n-1} M) = a^n M = 0$ , we have  $a^{n-1} M \subseteq \text{ann}_M(a) = 0$  and so  $a^{n-1} M = 0$  which is a contradiction. So that  $a \notin \sqrt{\text{Ann}_R(M)}$ . As  $N$  is an  $n$ -submodule and  $am \in N$ , we get  $m \in N$ . Hence,  $N$  is an  $r$ -submodule of  $M$ .  $\square$

**Theorem 2.22.** *Let  $N$  be a submodule of  $M$  such that  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$ . Then the following statements are equivalent:*

- (i)  $N$  is an  $n$ -submodule;
- (ii)  $N$  is a primary submodule of  $M$ .

*Proof.* (i)  $\Rightarrow$  (ii) Let  $ax \in N$  with  $a \notin \sqrt{(N :_R M)}$ , for  $a \in R$  and  $x \in M$ . As  $N$  is an  $n$ -submodule, we have  $x \in N$ . Thus  $N$  is a primary submodule.

(ii)  $\Rightarrow$  (i) Let  $ax \in N$  with  $a \notin \sqrt{\text{Ann}_R(M)}$ , for  $a \in R$  and  $x \in M$ . As  $\sqrt{(N :_R M)} = \sqrt{\text{Ann}_R(M)}$ , we have  $a \notin \sqrt{\text{Ann}_R(M)}$ . Since  $N$  is a primary submodule, we get  $x \in N$ . Therefore  $N$  is an  $n$ -submodule.  $\square$

By the proof of previous theorem, every  $n$ -submodule is a primary submodule. So it is straightforward to get that if  $N$  is an  $n$ -submodule of  $R$ -module  $M$ , then  $(N :_R M)$  is a primary ideal of  $R$ . Recall if  $(N :_R M)$  is a maximal ideal of ring  $R$ , then  $N$  is a primary submodule of  $M$ . So we have:

**Corollary 2.23.** *Let  $\text{Ann}_R(M)$  be a maximal ideal of  $R$ . Then every proper submodule of  $M$  is an  $n$ -submodule.*

By using the fact that every irreducible submodule of a Noetherian module is a primary submodule (Proposition 1-17 in [4]), we can get the following corollary:

**Corollary 2.24.** *Let  $M$  be a Noetherian  $R$ -module and  $N$  be an irreducible submodule of  $M$  such that  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$ . Then  $N$  is an  $n$ -submodule of  $M$ .*

**Proposition 2.25.** *If  $N$  is a primary  $R$ -submodule of  $M$  such that  $(N :_R M)$  is maximal in the set of all  $n$ -ideals, then  $N$  is an  $n$ -submodule of  $M$ .*

*Proof.* Let  $ax \in N$  with  $a \notin \sqrt{\text{Ann}_R(M)}$ , for  $a \in R$  and  $x \in M$ . By Theorem 2.11 [12],  $\sqrt{0} = \sqrt{(N :_R M)}$ . Since  $N$  is a primary submodule and  $a \notin \sqrt{(N :_R M)}$ ,  $x \in N$ . □

**Lemma 2.26.** *If  $N$  is an  $n$ -submodule and  $L$  is a primary submodule of an  $R$ -module  $M$  such that  $(L :_R M) \subseteq \text{Ann}_R(M)$ , then  $N \cap L$  is an  $n$ -submodule of  $M$ .*

*Proof.* Let  $rx \in N \cap L$  where  $r \notin \sqrt{\text{Ann}_R(M)}$ , for  $r \in R$ ,  $x \in M$ . Then  $r \notin \sqrt{(L :_R M)}$ . Since  $L$  is primary,  $x \in L$ . Also, since  $N$  is an  $n$ -submodule,  $x \in N$ . Thus  $x \in N \cap L$ . □

Recall that a proper ideal  $I$  of  $R$  is called semiprime, if whenever  $a^n \in I$  for  $a \in R$  and  $n \in \mathbb{N}$ , then  $a \in I$  [10]. Now, in the following theorem we give a characterization for torsion free modules in terms of  $n$ -submodules.

**Theorem 2.27.** *Let  $M$  be an  $R$ -module. Then the following statements are equivalent:*

- (i)  $M$  is a torsionfree  $R$ -module;
- (ii)  $M$  is faithful, zero submodule is an  $n$ -submodule of  $M$  and zero ideal is a semiprime ideal of  $R$ .

*Proof.* (i)  $\Rightarrow$  (ii) It follows from Lemma 2.5.  
 (ii)  $\Rightarrow$  (i) Let  $rx = 0$  and  $r \neq 0$ , for  $r \in R$ ,  $x \in M$ . Since (0) is a semiprime ideal of  $R$ ,  $\sqrt{0} = 0$ . As  $M$  is faithful, it follows that  $r \notin \sqrt{\text{Ann}_R(M)} = \sqrt{0} = 0$ . Since the zero submodule is an  $n$ -submodule,  $x = 0$ . Therefore,  $M$  is a torsion-free module. □

**Theorem 2.28.** *Let  $f : M \rightarrow M'$  be an  $R$ -homomorphism. Then the followings hold:*

- (i) If  $f$  is an epimorphism and  $N$  is an  $n$ -submodule of  $M$  containing  $\ker(f)$ , then  $f(N)$  is an  $n$ -submodule of  $M'$ .
- (ii) If  $f$  is a monomorphism and  $L'$  is an  $n$ -submodule of  $M'$ , then  $f^{-1}(L') = M$  or  $f^{-1}(L')$  is an  $n$ -submodule of  $M$ .

*Proof.* (i) Let  $rx' \in f(N)$  where  $r \notin \sqrt{\text{Ann}_R(M')}$ , for  $r \in R$ ,  $x' \in M'$ . Since  $f$  is epimorphism, there exists  $x \in M$  such that  $x' = f(x)$ . Then  $rx' = rf(x) = f(rx) \in f(N)$ . As  $\ker(f) \subseteq N$ , we conclude that  $rx \in N$ . Also, note that  $r \notin \sqrt{\text{Ann}_R(M)}$ . Since  $N$  is an  $n$ -submodule of  $M$ , we get the result that  $x \in N$  and so  $x' = f(x) \in f(N)$ .

(ii) Let  $f^{-1}(L') \neq M$  and  $rx \in f^{-1}(L')$  where  $r \notin \sqrt{\text{Ann}_R(M)}$ , for  $r \in R$ ,  $x \in M$ . Then  $f(rx) = rf(x) \in L'$ . Since  $f$  is a monomorphism and  $r \notin \sqrt{\text{Ann}_R(M)}$ , we get  $r \notin \sqrt{\text{Ann}_R(M')}$ . Since  $L'$  is an  $n$ -submodule of  $M'$ ,  $f(x) \in L'$  and so  $x \in f^{-1}(L')$ . Consequently,  $f^{-1}(L')$  is an  $n$ -submodule of  $M$ .  $\square$

**Corollary 2.29.** *Let  $M$  be an  $R$ -module and  $L \subseteq N$  be two submodules of  $M$ . Then the followings hold:*

- (i) *If  $N$  is an  $n$ -submodule of  $M$ , then  $\frac{N}{L}$  is an  $n$ -submodule of  $\frac{M}{L}$ .*
- (ii) *If  $\frac{N}{L}$  is an  $n$ -submodule of  $\frac{M}{L}$  and  $(L :_R M) \subseteq \sqrt{\text{Ann}_R(M)}$ , then  $N$  is an  $n$ -submodule of  $M$ .*
- (iii) *If  $\frac{N}{L}$  is an  $n$ -submodule of  $\frac{M}{L}$  and  $L$  is an  $n$ -submodule of  $M$ , then  $N$  is an  $n$ -submodule of  $M$ .*

*Proof.* (i) Assume that  $N$  is an  $n$ -submodule of  $M$  and  $L \subseteq N$ . Let  $\pi : M \rightarrow \frac{M}{L}$  be the natural homomorphism. Note that  $\ker(\pi) = L \subseteq N$ , and so by Theorem 2.28(i),  $\frac{N}{L}$  is an  $n$ -submodule of  $\frac{M}{L}$ .

(ii) Let  $rx \in N$  where  $r \notin \sqrt{\text{Ann}_R(M)}$  for  $r \in R$ ,  $x \in M$ . Then we have  $(r+I)(x+L) = rx+L \in \frac{N}{L}$  and  $r+I \notin \sqrt{\text{Ann}_{\frac{R}{I}}(\frac{M}{L})}$ , where  $I = (L :_R M)$ . Since  $\frac{N}{L}$  is an  $n$ -submodule of  $\frac{M}{L}$ , we conclude that  $x+L \in \frac{N}{L}$  and so  $x \in N$ . Consequently,  $N$  is an  $n$ -submodule of  $M$ .

(iii) It follows from (ii) and Proposition 2.3(i).  $\square$

**Corollary 2.30.** *Let  $M$  be an  $R$ -module and  $N$  be a submodule of  $M$ . If  $L$  is an  $n$ -submodule of  $M$  such that  $N \not\subseteq L$ , then  $L \cap N$  is an  $n$ -submodule of  $N$ .*

*Proof.* Consider the injection  $i : N \rightarrow M$ . Note that  $i^{-1}(L) = L \cap N$ , so by Theorem 2.28(ii),  $L \cap N$  is an  $n$ -submodule of  $N$ .  $\square$

Let  $M$  be an  $R$ -module and  $S$  be a multiplicative closed subset of  $R$ . Consider the natural homomorphism  $\pi$  from  $M$  to  $M_S$  as  $\pi(m) = \frac{m}{1}$ , for any  $m \in M$ . Then for each submodule  $L$  of  $M_S$ , we define  $L^c$  as an inverse image of  $L$  under this natural homomorphism.

**Proposition 2.31.** *Let  $M$  be an  $R$ -module and  $S$  a multiplicative closed subset of  $R$ .*

- (i) *If  $N$  is an  $n$ -submodule of  $M$ , then  $N_S = M_S$  or  $N_S$  is an  $n$ -submodule of  $M_S$ .*
- (ii) *If  $M$  is finitely generated,  $L$  is an  $n$ -submodule of  $M_S$  and  $S \cap (\text{Ann}_R(M) :_R a) = \emptyset$  for every  $a \notin \text{Ann}_R(M)$ , then  $L^c = M$  or  $L^c$  is an  $n$ -submodule of  $M$ .*

*Proof.* (i) Let  $N_S \neq M_S$  and  $\frac{a}{s} \frac{m}{t} \in N_S$  where  $\frac{a}{s} \notin \sqrt{\text{Ann}_{R_S}(M_S)}$ , for  $a \in R$ ,  $s, t \in S$ ,  $m \in M$ . Then we have  $uam \in N$ , for some  $u \in S$ . It is clear that  $a \notin \sqrt{\text{Ann}_R(M)}$ . Since  $N$  is an  $n$ -submodule of  $M$ , we conclude that  $um \in N$  and so  $\frac{m}{t} = \frac{um}{ut} \in N_S$ . Therefore  $N_S$  is an  $n$ -submodule of  $M_S$ .



(ii) Let  $L^c \neq M$  and  $am \in L^c$  where  $a \notin \sqrt{\text{Ann}_R(M)}$  for  $a \in R, m \in M$ . Then we have  $\frac{a}{1} \frac{m}{1} \in L$ . Now we show that  $\frac{a}{1} \notin \sqrt{\text{Ann}_{R_S}(M_S)}$ . Suppose  $\frac{a}{1} \in \sqrt{\text{Ann}_{R_S}(M_S)}$ . There exists a positive integer  $k$  such that  $(\frac{a}{1})^k M_S = 0$ . Then we get  $ua^k M = 0$  for some  $u \in S$ , as  $M$  is finitely generated. Since  $a \notin \sqrt{\text{Ann}_R(M)}$ ,  $a^k M \neq 0$  and so  $u \in (\text{Ann}_R(M) :_R a^k) \cap S$ , which is a contradiction. Thus we have  $\frac{a}{1} \notin \sqrt{\text{Ann}_{R_S}(M_S)}$ . As  $L$  is an n-submodule of  $M_S$ , we conclude that  $\frac{m}{1} \in L$  and so  $m \in L^c$ .  $\square$

**Lemma 2.32.** *Let  $M$  be a finitely generated  $R$ -module such that for every multiplicative closed set  $S \subseteq R$ , the kernel of  $\varphi : M \rightarrow M_S$  is either  $(0)$  or  $M$ . Then  $(0)$  is an n-submodule of  $M$ .*

*Proof.* Let  $rx = 0$  where  $r \in R - \sqrt{\text{Ann}_R(M)}$  and  $x \in M$ . So  $r^n \neq 0$ , for every  $n \in \mathbb{N}$ . We put  $S = \{r^n : n \in \mathbb{N} \cup \{0\}\}$ . Clearly  $S$  is a multiplicative closed set in  $R$ . If  $\ker(\varphi) = 0$ , then as  $\varphi(x) = \frac{x}{1} = \frac{rx}{r} = 0$  we have  $x = 0$ . Let  $\ker(\varphi) = M$ . Since  $M$  is finitely generated, we can write  $M = Rx_1 + Rx_2 + \dots + Rx_t$ , for some  $x_1, x_2, \dots, x_t \in M$ . Then  $\varphi(x_i) = \frac{x_i}{1} = 0$  for any  $1 \leq i \leq t$ . Thus for any  $i$ , there exists  $l_i \in \mathbb{N}$  such that  $r^{l_i} x_i = 0$ . Put  $j := \max\{l_1, l_2, \dots, l_t\}$ . Thus we have  $r^j M = 0$  and so  $r \in \sqrt{\text{Ann}_R(M)}$ , which is a contradiction.  $\square$

We recall that a nonempty subset  $S$  of  $R$  where  $R - \sqrt{0} \subseteq S$  is said to be an n-multiplicatively closed subset of  $R$ , if  $xy \in S$  for all  $x \in R - \sqrt{0}$  and all  $y \in S$  (see [12]).

**Theorem 2.33.** *Let  $M$  be a finitely generated  $R$ -module and  $N$  be a proper submodule of  $M$  such that  $(N :_R M) \cap S = \emptyset$ , where  $S$  is an n-multiplicatively closed set in  $R$ . Then there exists an n-submodule  $L$  of  $M$  containing  $N$  such that  $(L :_R M) \cap S = \emptyset$ .*

*Proof.* Consider that set  $\Omega = \{L : L \text{ is a submodule of } M; (L :_R M) \cap S = \emptyset\}$ . Since  $N \in \Omega$ , we have  $\Omega \neq \emptyset$ . Since  $M$  is finitely generated, by using Zorn's lemma, we get a maximal element  $K$  of  $\Omega$ . Now we show that  $K$  is an n-submodule of  $M$ . Suppose that  $rx \in K$ , for some  $r \notin \sqrt{\text{Ann}_R(M)}$  and  $x \notin K$ . Thus we get  $x \in (K :_M r)$  and  $K \subset (K :_M r)$ . By maximality of  $K$ , we have  $((K :_M r) :_R M) \cap S \neq \emptyset$  and thus there exists  $t \in S$  such that  $tM \subseteq (K :_M r)$ . Also  $rt \in S$ , because  $r \in R - \sqrt{0}$  and  $t \in S$  and  $S$  is an n-multiplicatively closed subset of  $R$ . We get  $(K :_R M) \cap S \neq \emptyset$ , which is a contradiction. Hence  $K$  is an n-submodule of  $M$ .  $\square$

**Proposition 2.34.** *Suppose that  $N \subseteq \bigcup_{i=1}^n N_i$ , where  $N, N_i$  ( $1 \leq i \leq n$ ), are  $R$ -submodules of  $M$ . If there exists  $N_j$  such that  $N \not\subseteq \bigcup_{i \neq j} N_i$ ,  $N_j$  is an n-submodule and  $(\bigcap_{i \neq j} N_i :_R M) \not\subseteq \sqrt{\text{Ann}_R(M)}$ , then  $N \subseteq N_j$ .*

*Proof.* We may assume that  $j = 1$ . Since  $N \not\subseteq \bigcup_{i \geq 2} N_i$ , there exists  $x \in N - \bigcup_{i=2}^n N_i$ . Thus we have  $x \in N_1$ . Let  $y \in N \cap (\bigcap_{i=2}^n N_i)$ . Since  $x \notin N_k$  and  $y \in N_k$  for every  $2 \leq k \leq n$ , we have  $x + y \notin N_k$ . Thus  $x + y \in N - \bigcup_{i=2}^n N_i$

and so  $x + y \in N_1$ . As  $x + y \in N_1$  and  $x \in N_1$ , it follows that  $y \in N_1$  and so  $N \cap (\bigcap_{i=2}^n N_i) \subseteq N_1$ . Also we have  $(\bigcap_{i=2}^n N_i :_R M)N \subseteq N \cap (\bigcap_{i=2}^n N_i)$ . Now since  $(\bigcap_{i=2}^n N_i :_R M)N \subseteq N_1$ ,  $(\bigcap_{i=2}^n N_i :_R M) \not\subseteq \sqrt{\text{Ann}_R(M)}$  and  $N_1$  is an n-submodule of  $M$ , we have  $N \subseteq N_1$ .  $\square$

Following Lemma 1.1 in [9], a submodule  $K$  of an  $R$ -module  $M$  is prime if and only if  $p = (K :_R M)$  is a prime ideal of  $R$  and the  $\frac{R}{p}$ -module  $\frac{M}{K}$  is torsion-free. Now, we give a similar result for n-submodules.

**Theorem 2.35.** *Let  $N$  be an  $R$ -submodule of  $M$  such that  $I = \sqrt{\text{Ann}_R(M)} \subseteq (N :_R M)$ . Then  $N$  is an n-submodule of  $M$  if and only if  $\frac{M}{N}$  is a torsion-free  $\frac{R}{I}$ -module.*

*Proof.* Let  $N$  be an n-submodule and  $(r + I)(x + N) = 0_{\frac{M}{N}}$ , for  $r \in R$  and  $x \in M$ . Then we have  $rx \in N$ . If  $r \in I$ , then  $r + I = 0$ . Otherwise, since  $N$  is an n-submodule, we conclude that  $x \in N$  and so  $x + N = 0$ . For the converse, assume that  $\frac{M}{N}$  is a torsion-free  $\frac{R}{I}$ -module and  $rx \in N$ , for  $x \in M$  and  $r \in R - \sqrt{\text{Ann}_R(M)}$ . Then  $(r + I)(x + N) = rx + N = N = 0_{\frac{M}{N}}$ . Now as  $\frac{M}{N}$  is a torsion-free  $\frac{R}{I}$ -module and  $r \notin I$ , we have  $x \in N$ . So  $N$  is an n-submodule of  $M$ .  $\square$

**Lemma 2.36.** *Let  $\{L_i\}_{i \in I}$  be a family of  $R$ -submodules of  $\{M_i\}_{i \in I}$ . If  $\prod_{i \in I} L_i$  is an n-submodule of  $\prod_{i \in I} M_i$ , then for every  $i \in I$ ,  $L_i$  is an n-submodule of  $M_i$ .*

*Proof.* Let  $\prod_{i \in I} L_i$  be an n-submodule of  $\prod_{i \in I} M_i$  and  $i$  be an arbitrary in  $I$ . We will prove  $L_i$  is an n-submodule of  $M_i$ . Suppose that  $rx \in L_i$  where  $r \notin \sqrt{\text{Ann}_R(M_i)}$ , for  $r \in R$  and  $x \in M_i$ . Put  $x_i := x$  and  $x_j := 0$  for all  $j \neq i$ . Then we have  $r(x_j)_{j \in I} \in \prod_{j \in I} L_j$  and  $r \notin \sqrt{\text{Ann}_R(\prod_{j \in I} M_j)}$ . Since  $\prod_{j \in I} L_j$  is an n-submodule of  $\prod_{j \in I} M_j$ , so  $(x_j)_{j \in I} \in \prod_{j \in I} L_j$ . Hence  $x_i \in L_i$ .  $\square$

**Corollary 2.37.** *Let  $M_1$  and  $M_2$  be  $R$ -module and  $M = M_1 \times M_2$ . Then the following are satisfied:*

- (i) *If  $L_1 \times M_2$  is an n-submodule of  $M$ , then  $L_1$  is an n-submodule of  $M_1$ .*
- (ii) *If  $M_1 \times L_2$  is an n-submodule of  $M$ , then  $L_2$  is an n-submodule of  $M_2$ .*

**Theorem 2.38.** *Let  $N$  be a proper  $R$ -submodule of  $M$ . Then  $N$  is an n-submodule of  $M$  if and only if for each  $a \in R - \sqrt{\text{Ann}_R(M)}$ , the homothety  $\lambda_a : \frac{M}{N} \rightarrow \frac{M}{N}$  is an injective.*

*Proof.* Suppose that  $N$  is an n-submodule and  $\lambda_a(x + N) = 0_{\frac{M}{N}}$  for  $a \in R - \sqrt{\text{Ann}_R(M)}$ ,  $x \in M$ . Then  $ax \in N$  and since  $N$  is an n-submodule, so  $x \in N$  and  $x + N = 0$ . Hence  $\lambda_a$  is injective. Conversely, suppose that  $rx \in N$  where  $r \notin \sqrt{\text{Ann}_R(M)}$ , for  $r \in R$ ,  $x \in M$ . It follows that  $\lambda_r(x + N) = 0$ . Since  $\lambda_r$  is injective,  $x + N = 0$  and so  $x \in N$ .  $\square$

In [7], I.G. Macdonald introduced the notion of secondary modules. A nonzero  $R$ -module  $M$  is said to be secondary, if for each  $a \in R$  the endomorphism of  $M$  given by multiplication by  $a$  is either surjective or nilpotent.

**Proposition 2.39.** *If  $M$  is a secondary  $R$ -module such that every ascending chain of cyclic submodules of it stops, then every proper submodule of  $M$  is an  $n$ -submodule.*

*Proof.* Let  $N$  be a proper submodule of  $M$  and  $rx \in N$ , for  $r \in R$  and  $x \in M$ . Assume that  $\varphi_r$  is the homothety  $M \rightarrow M$  for  $r \in R$ . If  $\varphi_r$  is nilpotent, then there exists  $n \in \mathbb{N}$  such that  $(\varphi_r)^n = 0$ . It follows that  $r^n \in \text{Ann}_R(M)$  and so  $r \in \sqrt{\text{Ann}_R(M)}$ . If  $\varphi_r$  is surjective, then we have

$$x = rx_1$$

$$x_1 = rx_2$$

$$x_2 = rx_3$$

...

$$x_n = rx_{n+1}$$

...

for some  $x_i \in M$ . Then  $\langle x \rangle \subseteq \langle x_1 \rangle \subseteq \langle x_2 \rangle \dots \subseteq \langle x_n \rangle \subseteq \dots$ . Since  $M$  is complete, there exists  $n \in \mathbb{N}$  such that  $\langle x_n \rangle = \langle x_i \rangle$ , for every  $i \geq n$ . Hence there exists  $s \in R$  such that  $x_{n+1} = sx_n$ . It follows that  $x_n = rsx_n$ . So  $(1 - rs)x = 0$  and we have  $x = sr x$ . As  $rx \in N$ , so  $x \in N$ .  $\square$

**Corollary 2.40.** *Let  $M$  be a Noetherian secondary module. Then every proper submodule is an  $n$ -submodule.*

**Proposition 2.41.** *If  $N$  is an  $n$ - $R$ -submodule of  $M$ , then  $N[x]$  an  $n$ -submodule of  $M[x]$ .*

*Proof.* Let  $r$  be a zero divisor of an  $R$ -module  $\frac{M[x]}{N[x]}$ . Since  $\frac{M[x]}{N[x]} \cong \frac{M}{N}[x]$ , then there exists  $f(x) = a_0 + a_1x + \dots + a_t x^t \in M[x]$  such that  $0 \notin \overline{f(x)} \in \frac{M}{N}[x]$  and  $r\overline{f(x)} = \overline{0}$ . Hence  $ra_i \in N$ , for  $1 \leq i \leq t$ . If for every  $i$ ,  $a_i \in N$ , then  $\overline{f(x)} = \overline{0}$ , which is a contradiction. Thus there exists  $1 \leq i \leq t$  such that  $a_i \notin N$  with  $ra_i \in N$ . On the other hand, as  $N$  is an  $n$ -submodule, so  $r \in \sqrt{\text{Ann}_R(M)}$ . Since  $M \subseteq M[x]$ , so  $r \in \sqrt{\text{Ann}_R(M[x])}$ . Then by Theorem 2.10,  $N[x]$  is an  $n$ -submodule of  $M[x]$ .  $\square$

## 3. EXAMPLES

EXAMPLE 3.1. Let  $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $R = \mathbb{Z}$ . Then every proper submodule of  $M$  is an n-submodule. It is clear that every proper submodule of  $M$  is prime and the colon ideal of  $M$  into submodules are equal  $2\mathbb{Z}$ . Now according to Proposition 2.13, every proper submodule of  $M$  is an n-submodule.

Now we have an example which shows that there exists an  $R$ -module that does not have an n-submodule.

EXAMPLE 3.2. Let  $p$  be any prime number. Let  $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$  and  $R = \mathbb{Z}$ . Then every proper submodule of  $M$  is not an n-submodule. Let  $N$  be an n-submodule of  $M$ . By Proposition 2.3(i),  $(N :_R M) \subseteq \sqrt{\text{Ann}_R(M)} = \sqrt{0} = 0$ . It follows that  $(N :_R M) = \sqrt{\text{Ann}_R(M)}$ . Then by Lemma 2.15,  $N$  is a prime submodule. On the other hand,  $pM$  is the only prime submodule of  $M$ . So  $N = pM$  and  $(N :_R M) = (pM :_R M) = p\mathbb{Z}$ , which is a contradiction.

*Remark 3.3.* (i) By Theorem 2.22, every n-submodule of a module is a primary submodule. However, the converse is not true in general. Since for example: if  $R = \mathbb{Z}$ ,  $M = \mathbb{Z}$  and  $N = 4\mathbb{Z}$ , then  $N$  is a primary submodule of  $M$ , however it is not n-submodule, as  $2 \cdot 2 \in N$ , but  $2 \notin \sqrt{\text{Ann}_R(M)}$  and  $2 \notin N$ .

(ii) It is well known that if  $N$  is a prime submodule of  $M$ , then  $(N :_R M)$  is a prime ideal of  $R$ . Contrary to what happens for a prime submodules, if  $N$  is an n-submodule, the ideal  $(N :_R M)$  is not in general an n-ideal of  $R$ . For example: Let  $M = \mathbb{Z}_4$ ,  $R = \mathbb{Z}$ . Take  $N = (\bar{0})$ . Certainly  $N$  is an n-submodule of  $M$ , but  $(N :_R M) = 4\mathbb{Z}$  is not an n-ideal of  $R$ .

The following example shows that the converse of Lemma 2.5, is not necessarily true.

EXAMPLE 3.4. Consider the  $\mathbb{Z}$ -module  $\mathbb{Z}_4$  and  $N = (\bar{0})$ . Clearly  $N$  is an n-submodule, but  $M$  is not a torsion-free module.

In the next example, we show that zero submodule is not always the only n-submodule of torsion-free modules.

EXAMPLE 3.5. Let  $M = \mathbb{Z} \oplus \mathbb{Z}$  and  $R = \mathbb{Z}$ . consider the submodule  $N = 0 \oplus \mathbb{Z}$ . Let  $a(m, n) = (am, an) \in N$  with  $a \notin \sqrt{\text{Ann}_R(M)} = 0$  for some  $a, m, n, \in \mathbb{Z}$ . Then we have  $am = 0$  and so  $m = 0$ . This implies that  $(m, n) = (0, n) \in N$ . Thus  $N$  is a nonzero n-submodule of  $M$ .

The next example shows that the sum of two n-submodule is not an n-submodule in general.

EXAMPLE 3.6. Let  $M = \mathbb{Z} \oplus \mathbb{Z}$  and  $R = \mathbb{Z}$ . Consider the submodules  $N = 0 \oplus \mathbb{Z}$  and  $K = \mathbb{Z} \oplus 0$ . One can easily see that  $K$  and  $N$  are n-submodules. Since  $N + K = M$ ,  $N + K$  is not an n-submodule of  $M$ .

**Proposition 3.7.**  $\mathbb{Q}$  as  $\mathbb{Z}$ -module has only one n-submodule.

*Proof.* By Lemma 2.5, zero submodule is an n-submodule of  $\mathbb{Q}$ . Let  $N$  be an n-submodule. It follows that  $(N :_{\mathbb{Z}} \mathbb{Q}) = 0$ . Then by Lemma 2.15,  $N$  is an prime submodule of  $\mathbb{Q}$ , which is zero.  $\square$

Now we give an example to show that in Theorem 2.27, it is necessary that zero submodule be an n-submodule.

EXAMPLE 3.8. Let  $M$  be the  $\mathbb{Z}$ -module  $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ .  $M$  is faithful and zero ideal is a semiprime ideal. By Example 3.2, zero submodule is not n-submodule of  $M$  and  $M$  is not torsion-free.

In the following examples we show that the condition  $\ker(f) \subseteq N$  in Theorem 2.28(i) and the condition monomorphism in Theorem 2.28(ii), are necessary.

EXAMPLE 3.9. Consider the  $\mathbb{Z}$ -epimorphism

$$\psi : \mathbb{Z} \longrightarrow \mathbb{Z}_6; \quad a \longmapsto \bar{a}$$

Clearly  $\psi(0) = \bar{0}$  and  $\ker(\psi) = 6\mathbb{Z} \not\subseteq (0)$ . By Example 2.1(ii),  $(\bar{0})$  is not n-submodule of  $\mathbb{Z}_6$ .

EXAMPLE 3.10. Consider the zero homomorphism

$$g : \mathbb{Q} \longrightarrow \mathbb{Z};$$

clearly  $\ker(g) = \mathbb{Q}$ . So  $g$  is not monomorphism. By Proposition 3.7,  $g^{-1}(0)$  is not an n-submodule.

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