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Groups whose Bipartite Divisor Graph for Character Degrees Has Five Vertices

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Abstract. Let G be a finite group and $\mathrm{cd}^*(G)$ be the set of nonlinear irreducible character degrees of G. Suppose that $\rho(G)$ denotes the set of primes dividing some element of $\mathrm{cd}^*(G)$. The bipartite divisor graph for the set of character degrees which is denoted by $B(G)$, is a bipartite graph whose vertices are the disjoint union of $\rho(G)$ and $\mathrm{cd}^*(G)$, and a vertex $p \in \rho(G)$ is connected to a vertex $a \in \mathrm{cd}^*(G)$ if and only if $p|a$. In this paper, we investigate the structure of a group G whose graph $B(G)$ has five vertices. Especially we show that all these groups are solvable.

Keywords: Bipartite divisor graph, Character degree, Solvable group.

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1. Introduction

Throughout this paper, G is a finite group. We write $\mathrm{cd}(G)$ to denote the set of irreducible character degrees of the group G, and we use $\mathrm{cd}^*(G)$ for the set $\mathrm{cd}(G) \setminus \{1\}$. Suppose that $\rho(G)$ is the set of primes dividing some element of $\mathrm{cd}^*(G)$. Exploring the interplay between the structure of a finite group G and the set $\mathrm{cd}(G)$ is a favorite research field in group theory. One of the questions that was studied extensively is the graphs attached to the set $\mathrm{cd}(G)$. A comprehensive survey on this topic can be found in [6]. The prime graph $\Delta(G)$ and the common divisor graph $\Gamma(G)$ are two important graphs associated to $\mathrm{cd}(G)$. The prime graph $\Delta(G)$ is the graph with vertex set $\rho(G)$

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and there is an edge between two vertices p and q if there exist some $n \in \mathrm{cd}(G)$ which is divisible by pq. The common-divisor graph $\Gamma(G)$ is the graph with vertex set $\mathrm{cd}^*(G)$ and two vertices m and n are connected if $\gcd(m,n) > 1$. In this paper we focus on bipartite divisor graph for $\mathrm{cd}^*(G)$. The bipartite divisor graph $B(G)$ is a bipartite graph whose vertices are the disjoint union of $\rho(G)$ and $\mathrm{cd}^*(G)$ and a vertex $p \in \rho(G)$ is connected to a vertex $a \in \mathrm{cd}^*(G)$ if and only if $p|a$. Groups whose $\Delta(G)$ or $\Gamma(G)$ has few vertices has been studied by many authors. For example, the prime graphs with four or fewer vertices are considered in papers [4, 8, 9]. In this note, we do an analogous work for bipartite divisor graph. The notion of $B(G)$ is introduced in [11] and groups whose $B(G)$ is a path or cycle are discussed in [2]. In this paper, we consider groups whose bipartite divisor graph has five vertices and obtain some group theoretical properties of these groups. We also provide examples of each possible graph.

2. Preliminaries

The following theorems will be used throughout the paper.

Theorem 2.1 (corollary 12.34 of [5]). *Let G be solvable. Then G has a normal abelian Sylow p-subgroup iff every element of* $\mathrm{cd}(G)$ *is relatively prime to p.*

Theorem 2.2 (corollary 12.2 of [5]). *Suppose $p|\chi(1)$ for every nonlinear $\chi \in$* $\mathrm{Irr}(G)$*, where p is a prime. Then G has a normal p-complement.*

Theorem 2.3 (Theorem 4.5 of [11]). *Let G be a group whose $B(G)$ is a complete bipartite graph. Then one of the following cases occurs:*

(a) *$G = AH$, where A is an abelian normal Hall subgroup of G and H is abelian, i.e. G is metabelian.*
(b) *$G = AH$, where A is an abelian normal Hall subgroup of G and H is a non-abelian p-group for some prime p. In particular, $\rho(G) = \{p\}$.*

Remark 2.4. Theorem 2.3 implies that the subgroup H is a $\rho(G)$-subgroup and A is a $\rho(G)'$-subgroup of G.

The following theorem from [12], helps us to obtain some examples of groups with a given set of character degrees.

Theorem 2.5 (Theorem 4.1 of [12]). *Let $1 < m_1 < \cdots < m_r$ be integers such that m_i divides m_{i+1} for all $i = 1, 2, \cdots, r-1$. Then there exists a group G with* $\mathrm{cd}(G) = \{1, m_1, \cdots, m_r\}$.

We also use the library of the small groups in GAP [1] for many examples and the kth group of order n in this library is recognizable by command SmallGroup(n, k) which is the symbol we use for this group.

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3. Groups whose $B(G)$ has five vertices

By definition of $\rho(G)$ and $\mathrm{cd}^*(G)$, each $p \in \rho(G)$ divides some element in $\mathrm{cd}^*(G)$ and every element $n \in \mathrm{cd}^*(G)$ is divisible by a prime number which lies in $\rho(G)$. Therefore, the graph $B(G)$ has no isolated vertex. Let G be a group whose graph $B(G)$ has five vertices. Using the fact that vertices of $B(G)$ are disjoint union of $\rho(G)$ and $\mathrm{cd}^*(G)$, one of the following cases occur:

(i) $|\rho(G)| = 1$ and $\mathrm{cd}^*(G) = 4$.
(ii) $|\rho(G)| = 4$ and $\mathrm{cd}^*(G) = 1$.
(iii) $|\rho(G)| = 2$ and $\mathrm{cd}^*(G) = 3$.
(iv) $|\rho(G)| = 3$ and $\mathrm{cd}^*(G) = 2$.

We investigate each case separately and determine the structure of groups of each possible case.

Theorem 3.1. *Let G be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 1$ then $G = AP$, where P is a Sylow p-subgroup for some prime p and A is a normal abelian p-complement.*

Proof. By hypothesis, $|\rho(G)| = 1$, hence $\mathrm{cd}(G) = \{1, p^a, p^b, p^c, p^d\}$ where p is a prime number. Since p divides the degree of every character, by Theorem 2.2, we conclude that G has a normal p-complement. Therefore $G = AP$ where P is a Sylow p-subgroup of G and A is a normal p-complement. In addition, every prime divisor of $|A|$ is coprime to p and by corollary 12.34 of [5], A is abelian and we are done. □

Theorem 3.2. *Let G be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 4$ then G' is abelian, $G' \cap Z(G) = 1$ and $G/Z(G)$ is a Frobenius group with cyclic complement.*

Proof. Since $B(G)$ has five vertices and $|\rho(G)| = 4$, therefore $\mathrm{cd}(G) = \{1, m\}$ where m is divisible by exactly four prime numbers. By corollary 12.6 of [5], G' is abelian. Assume that G is nilpotent. Since $|\mathrm{cd}(G)| = 2$, G is nonabelian. Let P be a nonabelian Sylow p-subgroup of G. If G has another nonabelian Sylow q-subgroup for some prime $q \neq p$, then $|\mathrm{cd}(G)| > 2$ which is a contradiction. Hence P is the only nonabelian Sylow subgroup of G which implies that every element of $\mathrm{cd}(G)$ must be a prime power which is a contradiction. Therefore G is not nilpotent. Now by using Theorem (C) of [3], we obtain the results. □

Example 3.3. Groups that satisfy hypothesis of Theorems 3.1 and 3.2 exist. For example, let P be a p-group of order p^3 with $\mathrm{cd}(P) = \{1, p\}$ and let $G_1 = P \times P \times P \times P$, then

$$\mathrm{cd}(G_1) = \{1, p, p^2, p^3, p^4\}.$$

Therefore G is a group that satisfy hypothesis of Theorem 3.1. Furthermore, if we replace P by the direct product of an abelian group A and the group P,

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then we have an example of Theorem 3.1 with non-trivial subgroup A.
For a group which satisfy Theorem 3.2, suppose that $F = GF(211)$ is a finite field with 211 elements and let H be the multiplicative group of F with 210 elements. Then H act Frobeniously on F and so the corresponding semidirect product $G_2 = FH$ is a Frobenius group with abelian kernel and complement. It is easy to check that $\mathrm{cd}(G_2) = \{1, 210\}$ and the results of Theorem 3.2 hold.

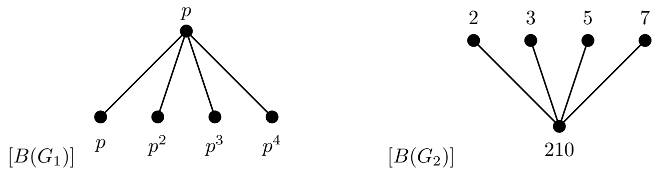

FIGURE 1. Graphs of example 3.3.

Theorem 3.4. *Let G be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 2$ then G is solvable and one of the following cases occurs:*

(i) *$G = HN$ where H is a Sylow p-subgroup of G or a Hall $\{p, q\}$-subgroup and N is a normal complement.*
(ii) *G is one of the families stated in* [7].

Proof. Since $B(G)$ has no isolated vertex, it is easy to check that $B(G)$ is one of the graphs in Figure 2.

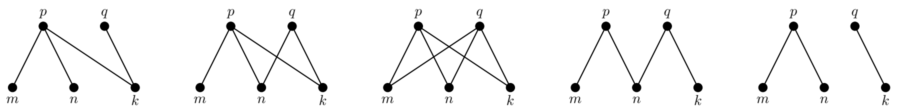

FIGURE 2. Possible graphs with $|\rho(G)| = 2$.

First, suppose that $B(G)$ is as graph (a) or (b) in Figure 2 where p and q are prime numbers and $\mathrm{cd}(G) = \{1, m, n, k\}$. Since p divides every nonlinear character degree of G, by Theorem 2.2, G has a normal p-complement, therefore $G = PN$ where P is a Sylow p-subgroup of G and N is a normal p-complement and case (i) occurs. We claim that in both cases G is solvable. Suppose that G is not solvable. Note that in both graphs (a) and (b) of Figure 2 there is a prime which divides every nonlinear character degree. Since $|\mathrm{cd}(G)| = 4$, using Theorem A and B of [10] we have $\mathrm{cd}(G) = \{1, r-1, r, r+1\}$ for some prime power r or $\mathrm{cd}(G) = \{1, 9, 10, 16\}$. In both cases there is no prime which divides every nonlinear character degree, thus graphs (a) and (b) can not occur

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for these groups. Therefore G is a solvable group. If $B(G)$ is as graph (c) in Figure 2, then $B(G)$ is a complete bipartite graph and by Theorem 2.3 case (i) holds. Furthermore, by corollary 4.2 of [11], G is solvable. If $B(G)$ is as graph (d) in Figure 2, then $B(G)$ is a path of length four and by proposition 2 of [2], G is a solvable group. Since for every prime $r \neq p, q$, r divides no character degree, using Theorem 2.1, we see that case (i) occurs. Now suppose that $B(G)$ is as graph (e) in Figure 2. Therefore $B(G)$ is a disconnected graph and has two connected components. We prove that G is solvable. Assume that G is not solvable. Again by Theorem A and B of [10], we must have $\mathrm{cd}(G) = \{1, r-1, r, r+1\}$ for some prime power r. Since $\mathrm{cd}(G) = \{1, p^a, p^b, q^c\}$ it follows that p divides two consecutive numbers which is impossible. Hence G is solvable and by Theorem 2.1 of [11], G belongs to a family of groups stated in [7] and case (ii) holds. □

The following example shows that all graphs in Figure 2 occur as $B(G)$ for some group G.

Example 3.5. Let $G_1 = \mathrm{SmallGroup}(108, 17)$, then $\mathrm{cd}(G_1) = \{1, 2, 4, 6\}$ and $B(G_1)$ is the same as graph (a) in Figure 2.
By Theorem 2.5, there is a group G_2 with $\mathrm{cd}(G_2) = \{1, 2, 6, 12\}$ and $B(G_2)$ is as graph (b) in Figure 2.
Suppose that $G_3 = \mathrm{SmallGroup}(108, 17)$, then $\mathrm{cd}(G_3) = \{1, 6, 12, 18\}$ and $B(G_3)$ is the graph (c) in Figure 2.
Assume that $G_4 = \mathrm{SmallGroup}(72, 15)$, then we have $\mathrm{cd}(G_4) = \{1, 2, 3, 6\}$ and $B(G_4)$ is the graph (d) in Figure 2.
Put $G_5 = \mathrm{SmallGroup}(48, 28)$, then $\mathrm{cd}(G_5) = \{1, 2, 3, 4\}$ and $B(G_5)$ is the graph (e) in Figure 2.

Theorem 3.6. *Let G be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 3$ then G is solvable and one of the following cases holds:*

(i) *$G = HN$ where H is a Sylow p-subgroup or a Hall $\{p, q\}$-subgroup or a Hall abelian $\{p, q, r\}$-subgroup of G and N is its normal complement.*
(ii) *$G = QN$ where Q is an abelian Sylow q-subgroup of G and N is its normal complement.*
(iii) *G is one of the families stated in* [7].

Proof. It's easy to verify that $B(G)$ is one of the graphs in Figure 3. Since $|\mathrm{cd}(G)| = 3$ Theorem 12.15 of [5] shows that in all cases G is a solvable group. Now we investigate each graph separately.

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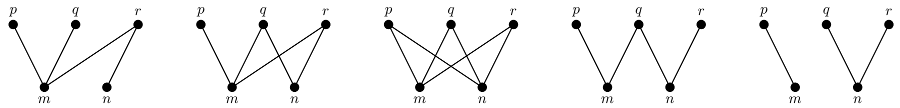

FIGURE 3. Possible graphs with $|\rho(G)| = 3$.

First suppose that $B(G)$ is as graph (a) in Figure 3. Then $\mathrm{cd}(G) = \{1, p^a q^b r^c, r^d\}$ and r divides every nonlinear character degree.Thus Theorem 2.2 implies that G has a normal r-complement. Therefore $G = HN$ where H is a Sylow r-subgroup of G and N is a normal r-complement and case (i) of theorem holds. Now assume that $B(G)$ is the graph (b) in Figure 3. In this case, we have $\mathrm{cd}(G) = \{1, p^a q^b r^c, q^d r^h\}$ and every nonlinear character degree is divisible by both q and r. Again Theorem 2.1 applies and $G = HN$ which H is a Hall $\{r, q\}$-subgroup and N is its normal complement. Hence case (i) occurs. If $B(G)$ is the graph (c) in Figure 3, then $B(G)$ is a complete graph. Applying Theorem 2.3, we have $G = HN$ where H is a Hall abelian $\rho(G)$-subgroup and N is an abelian normal complement of H. Therefore again case (i) of theorem holds. Now suppose that $B(G)$ is the graph (d) in Figure 3. Then $\mathrm{cd}(G) = \{1, p^a q^b, q^c r^d\}$. Since q divides every nonlinear character degree, G has a normal q-complement. Therefore, $G = QN$ where Q is a Sylow q-subgroup and N is its normal complement. Since $Q \cong G/N$ and $\mathrm{cd}(G)$ contains no powers of q, therefore Q is abelian. Hence case (ii) of theorem holds. Finally, suppose that $B(G)$ is the graph (e) in Figure 3. Then $B(G)$ is disconnected and has two connected components. Since G is solvable, Theorem 2.1 of [11] implies that G belongs to a family of groups stated in [7] and case (iii) holds. $\square$

EXAMPLE 3.7. We show that all graphs in Figure 3 really occur as $B(G)$ for some group G.
By Theorem 2.5 there exists a group G_1 with $\mathrm{cd}(G_1) = \{1, 5, 30\}$ and the graph $B(G_1)$ is the graph (a) in Figure 3.
Since $15|30|60$, Theorem 2.5 implies that there is a group G_2 with $\mathrm{cd}(G_2) = \{1, 15, 30\}$ and there is a group G_3 with $\mathrm{cd}(G) = \{1, 30, 60\}$. Thus the graphs $B(G_2)$ and $B(G_3)$ are the graphs (b) and (c) in Figure 3, respectively.
Suppose that $G_4 = \mathrm{SmallGroup}(960, 5748)$, then $\mathrm{cd}(G_4) = \{1, 12, 15\}$ and $B(G_4)$ is the graph (d) in Figure 3.
Let $G_5 = \mathrm{SmallGroup}(480, 1188)$, then $\mathrm{cd}(G_5) = \{1, 2, 15\}$ and the graph $B(G_5)$ is the graph (e) in Figure 3.

Suppose that G is a group which satisfy hypothesis of Theorem 3.6, then $\mathrm{cd}(G) = \{1, m, n\}$. If we apply Theorems of [12], then we can obtain more information about the structure of G, depending on the relation between $\pi(m)$

and $\pi(n)$, where $\pi(l)$ denotes the prime divisors of l. For details see [12].

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