On Bernstein Type Inequalities for Complex Polynomial

Elahe Khojastehnezhad, Mahmood Bidkham*

Department of Mathematics, University of Semnan, Semnan, Iran

E-mail: ekhojastehnejadelehe@semnan.ac.ir
E-mail: mbidkham@semnan.ac.ir; mdbidkham@gmail.com

Abstract. In this paper, we establish some Bernstein type inequalities for the complex polynomial. Our results constitute generalizations and refinements of some well-known polynomial inequalities.

Keywords: Inequality, Polynomial, Derivative, Maximum modulus, Restricted zeros.


1. Introduction and Statement of Results

Let $p$ be a polynomial of degree at most $n$. Then, according to a famous result known as Bernstein's inequality [8]

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|,$$

(1.1)

whereas concerning the maximum modulus of $p$ on a large circle $|z| = R > 1$, we have [20]

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|.$$

(1.2)

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequalities (1.1) and (1.2) can be sharpened. In fact, if $p(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can respectively be replaced by

*Corresponding Author

Received 11 August 2018; Accepted 16 August 2020
©2022 Academic Center for Education, Culture and Research TMU
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\] (1.3)

and

\[
\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|, \quad R > 1.
\] (1.4)

Inequality (1.3) was conjectured by Erdős and later verified by Lax [19], whereas Ankeny and Rivlin [5] used (1.3) to prove (1.4).

In the literature, there are already various generalizations and refinements of (1.3) and (1.4), for example (see Aziz [6], Bidkham et al. [9, 10, 11], Khojastehnezhad and Bidkham [17], Zireh [21], etc).

Inequalities (1.3) and (1.4) were sharpened by Dewan et.al [12, 13] proving that under the same hypothesis, for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > 1 \) and \( |z| = 1 \), we have

\[
|zp'(z) + \frac{n\beta}{2} p(z)| \leq \frac{n}{2} ((1 + \frac{\beta}{2}) \max_{|z|=1} |p(z)| - (1 + \frac{\beta}{2} - \frac{\beta}{2}) \min_{|z|=1} |p(z)|),
\] (1.5)

and

\[
|p(Rz) + \beta(R + \frac{1}{2})^n p(z)| \leq \frac{1}{2} (((R^n + \beta(R + \frac{1}{2})^n) + (1 + \beta(R + \frac{1}{2})^n)) \max_{|z|=1} |p(z)| - (R^n + \beta(R + \frac{1}{2})^n - 1 + \beta(R + \frac{1}{2})^n) \min_{|z|=1} |p(z)|).
\] (1.6)

Also they [12] proved if \( p \) has all its zeros in \( |z| \leq k \), \( k > 0 \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), we have

\[
\min_{|z|=1} |zp'(z) + \frac{n\beta}{2} p(z)| \geq n\left|1 + \frac{\beta}{2}\right| \min_{|z|=1} |p(z)|.
\] (1.7)

In this paper, we first prove an interesting result which is a compact generalization of inequality (1.7).

**Theorem 1.1.** If \( p \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), \( k > 0 \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \), \( R \geq r, rR \geq k^2 \) and \( |z| = 1 \), we have

\[
|p(Rz) - \alpha p(rz) + \beta(\frac{R+k}{r+k})^n p(rz)| \geq \frac{1}{k^n} |R^n - \alpha r^n + \beta(\frac{R+k}{r+k})^n - |\alpha| r^n| \min_{|z|=k} |p(z)|.
\] (1.8)

Assuming \( \alpha = 1 \) in Theorem 1.1, we have the following result.
Corollary 1.2. Let $p$ be a polynomial of degree $n$ such that does not vanish in $|z| > k$, $k > 0$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r$, $rR \geq k^2$ and $|z| = 1$, we get

$$|p(Rz) - p(rz) + \beta\{(R + k)^n - 1\}p(rz)| \geq \frac{1}{k^n}|R^n - r^n + \beta\{(R + k)^n - 1\}r^n|\min_{|z|=k}|p(z)|.$$  \hspace{1cm} (1.9)

By dividing the two sides of the inequality (1.9) by $(R-r)$ and letting $R \to r$, we get the following interesting result.

Corollary 1.3. Let $p$ be a polynomial of degree $n$ such that does not vanish in $|z| > k$, $k > 0$. Then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $r \geq k$ and $|z| = 1$, we get

$$|zp'(rz) + \frac{n\beta}{r + k}p(rz)| \geq \frac{n}{k^n}|r^{n-1} + \frac{\beta}{r + k}r^n|\min_{|z|=k}|p(z)|.$$ \hspace{1cm} (1.10)

Assuming $k = 1$, $r = 1$ in Corollary (1.3), we have the inequality (1.7). Using Theorem 1.1, we prove the following theorem, which provides a compact generalization of inequalities (1.5), (1.6).

Theorem 1.4. Let $p$ be a polynomial of degree $n$ such that it does not vanish in $|z| < k$, $k > 0$. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r$, $rR \geq \frac{1}{k^2}$ and $|z| = 1$,

$$|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk + 1}{r k + 1})^n - |\alpha|\}p(rk^2z)| \leq \frac{1}{2}\left\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{r k + 1})^n - |\alpha|\}r^n|\left|1 - \alpha + \beta\{(\frac{Rk + 1}{r k + 1})^n - |\alpha|\}\right|\max_{|z|=k}|p(z)| - \right. \hspace{1cm} (1.11)

\left. |k^n|R^n - r^n + \beta\{(\frac{Rk + 1}{r k + 1})^n - |\alpha|\}r^n| - |1 - \alpha + \beta\{(\frac{Rk + 1}{r k + 1})^n - |\alpha|\}\right|\min_{|z|=k}|p(z)|\right\}.$$

Equality holds for the polynomials $az^n + bk^n$, $|a| = |b|$.

Assuming $\alpha = 1$ in Theorem 1.4, we have the following result.

Corollary 1.5. Let $p$ be a polynomial of degree $n$ such that does not vanish in $|z| < k$, $k > 0$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R \geq r$, $rR \geq \frac{1}{k^2}$ and $|z| = 1$, we get

$$|p(Rk^2z) - p(rk^2z) + \beta\{(\frac{Rk + 1}{r k + 1})^n - 1\}p(rk^2z)| \leq \frac{1}{2}\left\{k^n|R^n - r^n + \beta\{(\frac{Rk + 1}{r k + 1})^n - 1\}r^n|\left|\max_{|z|=k}|p(z)| - \right. \hspace{1cm} (1.12)

\left. |k^n|R^n - r^n + \beta\{(\frac{Rk + 1}{r k + 1})^n - 1\}r^n| - |\beta\{(\frac{Rk + 1}{r k + 1})^n - 1\}|\min_{|z|=k}|p(z)|\right\}.$$
By dividing the two sides of the inequality (1.12) by \((R - r)\) and letting \(R \to r\), we get the following interesting result.

**Corollary 1.6.** Let \(p\) be a polynomial of degree \(n\) such that does not vanish in \(|z| < k, k > 0\). Then for every \(\beta \in \mathbb{C}\) with \(|\beta| \leq 1\), \(r \geq \frac{1}{k}\) and \(|z| = 1\), we have

\[
\left| k^2 z \frac{p'(rk^2 z)}{r k + 1} + \frac{n \beta k}{r k + 1} p(rk^2 z) \right| \leq \frac{n}{2} \left\{ \left| k^n |p^{n-1}| + \frac{\beta k}{r k + 1} r^n \right| + \left| \frac{-\beta k}{r k + 1} \right| \min_{|z|=k} |p(z)| - \left| \frac{\beta k}{r k + 1} \right| \max_{|z|=k} |p(z)| \right\}.
\]

(1.13)

**Remark 1.7.** Assuming \(k = 1\) and \(r = 1\) in Corollary 1.6 we have the inequality (1.5).

## 2. Lemmas

To prove of these theorems, we need the following lemmas. The first lemma is due to Aziz and Zargar [7].

**Lemma 2.1.** Let \(p\) be a polynomial of degree \(n\) having all its zeros in \(|z| \leq k\), \(k > 0\). Then for every \(R \geq r\) and \(r R \geq k^2\), we have

\[
|p(Rz)| \geq \left( \frac{R + k}{r + k} \right)^n |p(rz)|, \quad |z| = 1.
\]

(2.1)

**Lemma 2.2.** Let \(p\) be a polynomial of degree \(n\) such that does not vanish in \(|z| < k, k > 0\), and \(q(z) = z^n p(z/k)\). Then for all \(\alpha, \beta \in \mathbb{C}\) with \(|\alpha| \leq 1\), \(|\beta| \leq 1\), \(R \geq r\), \(r R \geq \frac{1}{k^2}\) and \(|z| = 1\), we have

\[
|p(Rk^2 z) - \alpha p(rk^2 z) + \beta \{ (\frac{Rk + 1}{r k + 1})^n - |\alpha| \} p(rk^2 z) | \leq k^n |q(Rz) - \alpha q(rz) + \beta \{ (\frac{Rk + 1}{r k + 1})^n - |\alpha| \} q(rz) |.
\]

(2.2)

**Proof.** Based on the hypotheses that the polynomial \(p\) has no zeros in \(|z| < k\), therefore the polynomial \(q(z) = z^n p(z/k)\) has all its zeros in \(|z| \leq \frac{1}{k}\). Since \(\frac{1}{k^n} |p(k^2 z)| = |q(z)|\) for \(|z| = \frac{1}{k}\), therefore the function \(\phi(z) = \frac{p(k^2 z)}{k^n q(z)}\) is analytic in the disc \(|z| \geq \frac{1}{k}\) and \(|\phi(z)| = 1\) on \(|z| = \frac{1}{k}\). Hence based on the the maximum modulus principle \(|\phi(z)| < 1\) for \(|z| > \frac{1}{k}\), or equivalently

\[
|p(k^2 z)| \leq k^n |q(z)|, \quad |z| \geq \frac{1}{k}.
\]

(2.3)

Since \(\frac{1}{k^n} |p(k^2 z)| = |q(z)|\) for \(|z| = \frac{1}{k}\), therefore for every real or complex number \(\delta\) with \(|\delta| < 1\) and \(|z| = \frac{1}{k}\), \(\delta p(k^2 z) < k^n q(z)\). Now using Rouche’s theorem it follows that all the zeros of \(H(z) := k^n q(z) + \delta p(k^2 z)\) lie in \(|z| \leq \frac{1}{k}\). While applying Lemma 2.1, we have

\[
|H(Rz)| \geq \left( \frac{Rk + 1}{r k + 1} \right)^n |H(rz) | > |H(rz)|, \quad |z| = 1,
\]

(2.4)
On Bernstein Type Inequalities for Complex Polynomial 115

where \( R > r, \ rR \geq \frac{1}{k^2} \).

It follows that for every \( \alpha \in \mathbb{C} \) with \( |\alpha| \leq 1 \), we get

\[
|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha||H(rz)| \geq \left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||H(rz)|, \ |z| = 1
\]
i.e.

\[
|H(Rz) - \alpha H(rz)| \geq \left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||H(rz)| \text{ for } |z| = 1. \tag{2.5}
\]

Since \( H(Rz) \) has all its zeros in \( |z| \leq \frac{1}{Rk} < 1 \), and \( |H(rz)| < |H(Rz)| \) for \( |z| = 1 \), a direct application of Rouche’s theorem shows that the polynomial \( H(Rz) - \alpha H(rz) \) has all its zeros in \( |z| < 1 \). Using Rouche’s theorem again, it follows that for every \( \beta \in \mathbb{C} \) with \( |\beta| < 1 \) and \( R > r, \ rR \geq \frac{1}{k^2} \), all the zeros of the polynomial

\[
T(z) = H(Rz) - \alpha H(rz) + \beta\left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||H(rz)|
\]
lie in \( |z| < 1 \).

Replacing \( H(z) \) by \( k^n q(z) + \delta p(k^2 z) \), we conclude that all the zeros of

\[
T(z) = k^n |q(Rz) - \alpha q(rz)| + \beta\left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||q(rz)| + \delta \left\{ \frac{p(Rk^2 z) - \alpha p(rk^2 z)}{r^k + 1} \right\}^n - \frac{R + k}{r + k}|p(rk^2 z)| \tag{2.6}
\]
lie in \( |z| < 1 \), for every \( R > r, \ rR \geq \frac{1}{k^2} \), \( |\alpha| \leq 1 \), \( |\beta| < 1 \) and \( |\delta| < 1 \). We now show that (2.6) implies (2.2). Indeed, suppose otherwise. Then, there is a point \( z = z_0 \) with \( |z_0| = 1 \) such that

\[
|p(Rk^2 z_0) - \alpha p(rk^2 z_0)| \geq k^n|q(Rz_0) - \alpha q(rz_0)| + \beta\left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||q(rz_0)|.
\]

We take

\[
\delta = -\frac{k^n|q(Rz_0) - \alpha q(rz_0)| + \beta\left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||q(rz_0)|}{p(Rk^2 z_0) - \alpha p(rk^2 z_0) + \beta\left\{ \frac{Rk + 1}{r^k + 1} \right\}^n - |\alpha||q(rz_0)|},
\]
then \( |\delta| < 1 \) and with this choice of \( \delta \), we have, \( T(z_0) = 0 \) for \( |z_0| = 1 \). But this contradicts that \( T \) has all its zeros in \( |z| < 1 \). For the case \( \beta \), with \( |\beta| = 1 \), (2.2) follows by continuity. For \( R = r \) inequality (2.2) follows by inequality (2.3). This completes the proof of Lemma 2.2.

\[\square\]

**Lemma 2.3.** Let \( p \) be a polynomial of degree \( n \). Then for all \( \alpha, \ \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \), \( R \geq r, \ rR \geq k^2 \), \( k > 0 \) and \( |z| = 1 \), we have

\[
|p(Rz) - \alpha p(rz) + \beta\left\{ \frac{R + k}{r + k} \right\}^n - |\alpha||p(rz)| \leq \frac{1}{k^n|R^n - \alpha r^n + \beta\left\{ \frac{R + k}{r + k} \right\}^n - |\alpha||r^n|\max_{|z| = k}|p(z)|. \tag{2.7}
\]
Proof. Let $M = \max_{|z|=k} |p(z)|$, then for $\delta$ with $|\delta| > 1$, we can conclude from Rouche’s theorem that all zeros of polynomial $H(z) = p(z) - \delta M(\frac{z}{k})^n$ lie in the closed disk $|z| \leq k$, $k > 0$. Using Lemma 2.1, we have

$$|H(Rz)| \geq \left( \frac{R+k}{r+k} \right)^n |H(rz)| > |H(rz)|, \quad |z| = 1,$$

(2.8)

where $R > r$, $rR \geq k^2$.

It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, we get

$$|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha| |H(rz)| \geq \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} |H(rz)|, \quad |z| = 1,$$

i.e.

$$|H(Rz) - \alpha H(rz)| \geq \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} |H(rz)|, \quad |z| = 1.$$

(2.9)

Since $H(Rz)$ has all its zeros in $|z| \leq \frac{k}{R} < 1$, and $|H(rz)| < |H(Rz)|$, a direct application of Rouche’s theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Using Rouche’s theorem again, implies that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $R > r$, $rR \geq k^2$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta \left( \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right) H(rz)$$

lie in $|z| < 1$.

Replacing $H(z)$ by $p(z) - \delta M(\frac{z}{k})^n$, we conclude that all the zeros of

$$T(z) = \left[ p(Rz) - \alpha p(rz) + \beta \left( \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right) p(rz) \right] +$$

$$\delta \frac{Mz^n}{k^n} \left\{ R^n - \alpha r^n + \beta \left( \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right) r^n \right\}$$

(2.10)

lie in $|z| < 1$, for every $R > r$, $rR \geq k^2$, $|\alpha| \leq 1$, $|\beta| < 1$ and $|\delta| > 1$. This implies

$$|p(Rz) - \alpha p(rz) + \beta \left( \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right) p(rz)| \leq$$

$$|R^n - \alpha r^n + \beta \left( \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right) r^n| \frac{M}{k^n},$$

(2.11)

where $|z| = 1$.

For $\beta$, with $|\beta| = 1$, (2.11) follows by continuity. For $R = r$ inequality (2.11) reduces to $|p(rz)| \leq \frac{k^n}{R^n} \max_{|z|=k} |p(z)|$ which it follows by taking $p(kz)$ and $|z| = \frac{k}{R}$, where $\frac{k}{R} \geq 1$ in inequality (1.2). This completes the proof of Lemma 2.3. □
Lemma 2.4. If \( p \) is a polynomial of degree \( n \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, \ |\beta| \leq 1, \ R \geq r, \ rR \geq \frac{1}{k^2} \) and \( |z| = 1 \),

\[
|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}p(rk^2z)| +
\]

\[
k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}q(rz)| \leq
\]

\[
\{k^n|R^n - \alpha r^n + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}| \max_{|z|=1} |p(z)|, \]

(2.12)

where \( q(z) = z^n p(\frac{1}{z^2}) \).

Proof. Assume that \( M = \max_{|z|=k} |p(z)| \). Then, for \( \delta > |\delta| > 1 \), we can conclude from Rouche’s theorem that the polynomial \( G(z) = p(z) - \delta M \) does not vanish in \( |z| < k \). If we take \( H(z) = z^n G(1/z) \), then \( |G(k^2z)| = k^n |H(z)| \) for \( |z| = \frac{1}{k} \).

Using Lemma 2.2, for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, \ |\beta| \leq 1, \ R \geq r, \ rR \geq \frac{1}{k^2} \) and \( |z| = 1 \), we have

\[
|G(Rk^2z) - \alpha G(rk^2z) + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}G(rk^2z)| \leq
\]

\[
k^n |H(Rz) - \alpha H(rz) + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}H(rz)|. \quad (2.13)
\]

Therefore, by using the equality

\[
H(z) = z^n G(\frac{1}{z}) = z^n p(\frac{1}{z}) - \delta M z^n
\]

\[
= q(z) - \delta M z^n,
\]

we get

\[
\{|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}p(rk^2z)| -
\]

\[
\delta\{1 - \alpha + \beta\{(\frac{R}{r+1})^n - |\alpha|\}\}M \leq
\]

\[
k^n\{|q(Rz) - \alpha q(rz) + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}q(rz)| -
\]

\[
\delta\{R^n - \alpha r^n + \beta\{(\frac{R}{rk+1})^n - |\alpha|\}r^n\}M|.
\]

(2.14)

Since \( \frac{1}{k^n} |p(k^2z)| = |q(z)| \) for \( |z| = \frac{1}{k} \), therefore

\[
\max_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \max_{|z|=k} |p(z)|,
\]

\[
\max_{|z|=\frac{1}{k}} |q(z)| = \frac{M}{k^n}.
\]
Now by applying Lemma 2.3 to \( q(z) \) for \( k > 0 \), we have

\[
|q(Rz) - \alpha q(rz) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|q(rz)| \leq \max_{|z|=\frac{1}{k}} |q(z)|.
\]

i.e.

\[
|q(Rz) - \alpha q(rz) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|q(rz)| \leq \{R^n - \alpha r^n + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|r^n\} M.
\]

Now by suitable choice of argument of \( \delta \), we get

\[
|q(Rz) - \alpha q(rz) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|q(rz)| = |\delta||R^n - \alpha r^n + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|r^n|M - \left|q(Rz) - \alpha q(rz) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|q(rz)\right|.
\]

Combining right hand sides of (2.14) and (2.15) we can obtain

\[
|p(Rk^2z) - \alpha p(rk^2z) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|p(rk^2z)| - |\delta||1 - \alpha + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha||M \leq |\delta||k^n|R^n - \alpha r^n + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|r^n|M| - k^n|q(Rz) - \alpha q(rz) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|q(rz)|,
\]

which implies

\[
|p(Rk^2z) - \alpha p(rk^2z) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|p(rk^2z)| + k^n|q(Rz) - \alpha q(rz) + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|q(rz)| \leq |\delta||k^n|R^n - \alpha r^n + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha|r^n| + |1 - \alpha + \beta\left(\frac{Rk + 1}{rk + 1}\right)^n - |\alpha||M|.
\]

Making \( |\delta| \to 1 \), we have the result. \( \Box \)

**Lemma 2.5.** Let \( p \) be a polynomial of degree \( n \) having no zeros in \( |z| < k \), \( k > 0 \). Then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \), \( R \geq r \), \( Rr \geq \frac{1}{k^2} \) and \( |z| = 1 \),
we have
\[ |p(R_k^2z) - \alpha p(r_k^2z) + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}p(r_k^2z)| \leq \frac{1}{2} \]
and
\[ \{k^n\}|R^n - \alpha r^n + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}r^n| \leq \max_{|z|=k}|p(z)| \]

(2.16)

Proof. Since \( p \) does not vanish in \( |z| < k, k > 0 \), Lemma 2.2, yields
\[ |p(R_k^2z) - \alpha p(r_k^2z) + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}p(r_k^2z)| \leq k^n|q(Rz) - \alpha q(rz) + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}q(rz)|, \]

(2.17)

Now by combining the inequalities (2.12) and (2.17), we have
\[ 2|p(R_k^2z) - \alpha p(r_k^2z) + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}p(r_k^2z)| \leq k^n|q(Rz) - \alpha q(rz) + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}q(rz)| \]
\[ \{k^n\}|R^n - \alpha r^n + \beta\{(R_k + 1/r_k + 1)^n - |\alpha|\}r^n| \leq \max_{|z|=k}|p(z)|. \]

(2.18)

This gives the result. \( \square \)

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. If \( p \) has a zero on \( |z| = k \), then inequality is trivial. Therefore, we assume that \( p(z) \) has all its zeros in \( |z| < k \). If \( m = \min_{|z|=k}|p(z)| \), then \( m > 0 \) and \( |p(z)| \geq m \) for \( |z| = k \). If \( |\lambda| < 1 \), then it follows by Rouche’s theorem that the polynomial \( p(z) - \lambda m(z/k)^n \), has all its zeros in \( |z| < k, k > 0 \). Proceeding similarly as in the proof of Lemma 2.3, it follows that all the zeros of
\[ p(Rz) - \alpha p(rz) + \beta\{(R + k/r + k)^n - |\alpha|\}p(rz) + \]
\[ \lambda m(z/k)^n\{R^n - \alpha r^n + \beta\{(R + k/r + k)^n - |\alpha|\}r^n\} \]

lie in \( |z| < 1 \), for every \( R \geq r, Rr \geq k^2, |\alpha| \leq 1, |\beta| < 1 \) and \( |\lambda| < 1 \). This implies

\[ \frac{m}{k^n}|R^n - \alpha r^n + \beta\{(R + k/r + k)^n - |\alpha|\}r^n| \leq \]
\[ |p(Rz) - \alpha p(rz) + \beta\{(R + k/r + k)^n - |\alpha|\}p(rz)|, \]

(3.1)
where $|z| = 1$. This completes the proof. □

Proof of Theorem 1.4. If $p(z)$ has a zero on $|z| = k$, then $\min_{|z|=k} |p(z)| = 0$ and in this case the result follows from Lemma 2.5. Hence we assume that $p(z) \neq 0$ in $|z| \leq k$. In this case we have $m = \min_{|z|=k} |p(z)| > 0$ and for $\gamma$ with $|\gamma| < 1$, we get $|\gamma m| < m \leq |p(z)|$, where $|z| = k$. Now we conclude from Rouche’s theorem that the polynomial $G(z) = p(z) - \gamma m$ does not vanish in $|z| < k$. If we take $H(z) = z^n G(1/z)$, then by using the polynomials $G(z)$ and $H(z)$ in Lemma 2.2, we have

$$|G(Rk^2 z) - \alpha G(rk^2 z) + \beta \{(Rk + 1)/rk + 1\} G(rk^2 z)| \leq \left| k^n |H(Rz) - \alpha H(rz) + \beta \{(Rk + 1)/rk + 1\} H(rz) | \right|.$$ (3.3)

Using the fact that

$$H(z) = z^n G(1/z) = z^n p(1/z) - \gamma m z^n = q(z) - \gamma m z^n,$$

or

$$H(z) = q(z) - \gamma m z^n,$$

and substituting $G(z)$ and $H(z)$ in (3.3), we get

$$\left| \left| \left\{ p(Rk^2 z) - \alpha p(rk^2 z) + \beta \left\{ \left( \frac{Rk + 1}{rk + 1} \right)^n - |\alpha| \right\} p(rk^2 z) \right| \right\| - \gamma \{ 1 - \alpha + \beta \left\{ \left( \frac{Rk + 1}{rk + 1} \right)^n - |\alpha| \right\} m \right\| \leq k^n |\{ q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk + 1}{rk + 1} \right)^n - |\alpha| \right\} q(rz) \}| - \gamma \left| R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk + 1}{rk + 1} \right)^n - |\alpha| \right\} r^n \right| m z^n \right|.$$ (3.4)

Since the polynomial $q(z) = z^n p(1/z)$ has all zeros in $|z| \leq 1/k$ and $m = \min_{|z|=1/k} |p(z)| = k^n \min_{|z|=1/k} |q(z)|$, hence by applying Theorem 1.1 for the polynomial $q(z)$ with $1/k$, we obtain

$$\left| R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk + 1}{rk + 1} \right)^n - |\alpha| \right\} r^n \right| m \leq \left| q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk + 1}{rk + 1} \right)^n - |\alpha| \right\} q(rz) \right|.$$
Therefore, by suitable choice of argument of $\gamma$, we get

$$
|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}q(rz)| - \\
\pi\{R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}r^n m| = \\
|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}q(rz)| - \\
|\gamma||R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}r^n m|.
$$

(3.5)

Now combining (3.4) and (3.5), we get

$$
|p(Rk^2 z) - \alpha p(rk^2 z) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}p(rk^2 z)| - \\
\kappa |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}q(rz)| - \\
|\gamma||\kappa |R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}r^n m|.
$$

This implies

$$
|p(Rk^2 z) - \alpha p(rk^2 z) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}p(rk^2 z)| \leq \\
\kappa |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}q(rz)| - \\
|\gamma||\kappa |R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}r^n m| - |1 - \alpha + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}|m.
$$

Letting $|\gamma| \to 1$, we have

$$
|p(Rk^2 z) - \alpha p(rk^2 z) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}p(rk^2 z)| \leq \\
\kappa |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}q(rz)| - \\
\{\kappa |R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}r^n | - |1 - \alpha + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}|m.
$$

(3.6)

On the other hand, based on Lemma 2.4, we have

$$
|p(Rk^2 z) - \alpha p(rk^2 z) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}p(rk^2 z)| + \\
\kappa |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}q(rz)| \leq \\
\{\kappa |R^n - \alpha r^n + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}r^n | + |1 - \alpha + \beta\{(\frac{Rk + 1}{rk + 1})^n - |\alpha|\}|m. \\
\max \|p(z)\|_{|z|=k}.
$$

(3.7)
Combining (3.6) and (3.7), we get (1.11) and this completes the proof of Theorem 1.4.

□

ACKNOWLEDGMENTS

The authors wish to thank the referees for their comments and suggestions.

REFERENCES
