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Quotient BCI-algebras induced by pseudo-valuations

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Abstract. In this paper, we study pseudo-valuations on a BCI-algebra and obtain some related results. The relation between pseudo-valuations and ideals is investigated. We use a pseudo-metric induced by a pseudo-valuation to introduce a congruence relation on a BCI-algebra. We define the quotient algebra induced by this relation and prove that it is also a BCI-algebra and study its properties.

Keywords: BCI-algebra, pseudo-valuation, ideal, pseudo-metric, quotient algebra.

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1. Introduction

The notions of BCK and BCI-algebras were introduced by Imai and Iseki in [7, 8]. They are two important classes of logical algebras. BCI-algebras are generalization of BCK-algebras. Some properties of these structures were presented in [1, 4, 6, 10, 11, 12] and [13]. Recently, D. Busneag [2, 3] introduced the notion of a pseudo valuation and applied it to Hilbert-algebras and residuated lattices. Also, M. I. Doh and M. S. Kang [5] applied pseudo valuations

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to BCK/BCI algebras and investigate some properties.
In the next section, some preliminary definitions and theorems are stated. In section 3, we study pseudo-valuation on BCI-algebras and investigate its properties which is not in [5]. We discuss the relation among pseudo-valuations and ideals of a BCI-algebra. We obtain some results of pseudo-metrics induced by pseudo-valuations on BCI-algebras and prove that a pseudo-metric induced by a pseudo-valuation υ is a metric on a BCK-algebra if and only if υ is a valuation but it may not be true in general for a BCI-algebra. In section 4, we use pseudo-metric induced by a pseudo-valuation to define the quotient algebra. We prove that this quotient algebra is also a BCI-algebra and obtain some related results.

2. Preliminaries

Definition 2.1.[11] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCI-algebra*, if it satisfies the following conditions: for any $x, y, z \in X$:
(BCI 1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI 2) $x * 0 = x$,
(BCI 3) $x * y = 0$ and $y * x = 0$ imply $x = y$.
We call the binary operation $*$ on X the *multiplication* on X and the constant of X the *zero element* of X. We often write X instead of $X = (X, *, 0)$ for a BCI-algebra in brevity.

Theorem 2.2.[11] Let X be a BCI-algebra. Define a binary relation $\leq$ on X by which $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$. Then $(X, \leq)$ is a partially ordered set with 0 is a minimal element in the meaning that $x \leq 0$ implies $x = 0$.

A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a *BCK-algebra*.[10]
The set of all positive elements of a BCI-algebra X is called the *BCK-part* of X and is denoted by $B(X)$.

Theorem 2.3.[10, 11] Let x, y, z be any elements in a BCI-algebra X. Then
(1) $x \leq y$ implies $z * y \leq z * x$,
(2) $x \leq y$ implies $x * z \leq y * z$,
(3) $x * y \leq z$ if and only if $x * z \leq y$,
(4) $x * (x * y) \leq y$,
(5) $(x * y) * (z * y) \leq (x * z)$,
(6) $(x * y) * (x * z) \leq (z * y)$,
(7) $(x * y) * z = (x * z) * y$,
(8) $x \leq x$,
(9) $0 * (x * y) = (0 * x) * (0 * y)$.

A subset Y of a BCI-algebra X is called a *subalgebra* of X if constant 0 of X is in Y, and $(Y, *, 0)$ itself forms a BCI-algebra. $B(X)$ is a subalgebra of a BCI-algebra X.

Definition 2.4.[11] A subset I of a BCI-algebra X is called an *ideal* of X if
(1) $0 \in I$,
(2) $y \in I$, $x * y \in I$ imply $x \in I$ for any $x, y \in X$.
Any ideal I has the property: $y \in I$ and $x \leq y$ imply $x \in I$.

Definition 2.5.[11] An ideal I of a BCI-algebra X is called *closed* if I is closed under $*$ on X (i.e, I is a subalgebra of X).

Proposition 2.6.[11] An ideal I of a BCI-algebra X is closed if and only if $0 * x \in I$ for any $x \in I$.

Proposition 2.7.[11] Let X be a BCI-algebra. Then
(i) If an ideal of X is a finite order, then it is closed, especially, if X is a finite order, then any ideal of X is closed.
(ii) If X is a BCK-algebra, then any ideal of X is closed.

Definition 2.8.[11] Let X and Y be BCI-algebras. A map $f : X \to Y$ is called *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

f is called *epimorphism*, if it is a surjective homomorphism. f is called *monomorphism*, if it is a injective homomorphism. An *isomorphism* means that f is both of epimorphic and monomorphic. Moreover, we say X is *isomorphic* to Y, symbolically, $X \cong Y$, if there is an isomorphism from X to Y. For a homomorphism $f : X \to Y$, we have $f(0) = \acute{0}$ where 0 and $\acute{0}$ are zero elements of X and Y, respectively.

Definition 2.9.[11] An equivalence relation θ on a BCI-algebra X is called a *congruence relation* on X, if $(x, y) \in \theta$ implies $(x*z, y*z) \in \theta$ and $(z*x, z*y) \in \theta$ for all $x, y, z \in X$.

Theorem 2.10.[11] Let I be an ideal of a BCI-algebra X. Define a binary relation θ_I on X as follows: $(x, y) \in \theta_I$ if and only if $x * y, y * x \in I$, for all $x, y \in X$. Then θ_I is a congruence relation on X which is called the *ideal congruence* on X induced by the ideal I.

Theorem 2.11.[11] Let I be an ideal of a BCI-algebra X and θ_I be the ideal congruence relation. The set of all equivalence classes $[x]_I = \{y \in X : (x, y) \in \theta_I\}$ is denoted by X/I. On this set, we define $[x]_I * [y]_I = [x * y]_I$. Then

$(X/I, *, [0]_I)$ is a BCI-algebra.

3. Pseudo-valuations on BCI-algebras

Definition 3.1.[4] A real function $\upsilon : X \to \Re$ is called a *pseudo-valuation* on a BCI-algebra X if it satisfies the following conditions:
(V1) $\upsilon(0) = 0$,
(V2) $\upsilon(x) \leq \upsilon(x * y) + \upsilon(y)$; for all $x, y \in X$.
The pseudo-valuation υ is said to be a *valuation* if
(V3) $\upsilon(x) = 0$ implies $x = 0$.

Example 3.2.(i) Let X be an arbitrary BCI-algebra and $c \in \Re$ such that $c \geq 0$. Define $\upsilon : X \to \Re$ by $\upsilon(x) = c$ for all $x \in X - \{0\}$ and $\upsilon(0) = 0$. Then υ is a pseudo-valuation on X. If $c = 0$, then υ is called *zero pseudo-valuation*.
(ii) The set Z of integer, together with the binary operation $*$ defined by $x * y = x - y$ forms a BCI-algebra, where the operation - is the subtraction as usual. Let $a \neq 0$ be an arbitrary element of Z. Then $\upsilon(x) = ax$ is a valuation on Z.

Theorem 3.3. Let υ be a pseudo-valuation on a BCI-algebra X. Then
(1) $x \leq y$ implies $\upsilon(x) \leq \upsilon(y)$,
(2) $\upsilon(x * y) \leq \upsilon(x * z) + \upsilon(z * y)$,
(3) $0 \leq \upsilon(x * y) + \upsilon(y * x)$,
for all $x, y, z \in X$.

Proof. See Proposition 3.11 in [4]. □

Corollary 3.4. Let υ be a pseudo-valuation on a BCI-algebra X. If $x \in B(X)$, then $\upsilon(x) \geq 0$.

Proof. Since $x \in B(X)$, then $0 \leq x$. By Theorem 3.3 part (1), we get that $0 = \upsilon(0) \leq \upsilon(x)$. □

In the following example, we will show that if υ is a pseudo-valuation on a BCI-algebra X such that $\upsilon(x) \geq 0$ where $x \in X$, then it may not be true $x \in B(X)$ in general.

Example 3.5. Let X be a BCI-algebra with the universe $\{0, 1, a\}$ such that the operation $*$ is defined by the table below:

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Define $\upsilon(0) = 0$, $\upsilon(1) = 3$ and $\upsilon(a) = 6$. Then υ is a pseudo-valuation on X and $\upsilon(a) \geq 0$. But we have $a \notin B(X)$.

Theorem 3.6. Let I be an ideal of a BCI-algebra X and t be a positive element of $\Re$. Define $\upsilon_I : X \to \Re$,

$$\upsilon_I(x) = \{ \begin{array}{ll} 0 & x \in I \\ t & x \notin I \end{array}$$

Then υ_I is a pseudo-valuation on X which is called the *pseudo-valuation induced by ideal* I. Moreover υ_I is a valuation if and only if $I = \{0\}$.

Proof. The proof is straightforward. □

Theorem 3.7. Let υ be a pseudo-valuation on a BCI- algebra X. Then $I_\upsilon = \{x \in X : \upsilon(x) \leq 0\}$ is an ideal of X which is called the *ideal induced by pseudo-valuation* υ.

Proof. Since $\upsilon(0) = 0$, we have $0 \in I_\upsilon$. Suppose that $y, x * y \in I_\upsilon$. Then $\upsilon(y), \upsilon(x * y) \leq 0$. We get that

$$\upsilon(x) \leq \upsilon(x * y) + \upsilon(y) \leq 0$$

Therefore $x \in I_\upsilon$ and I_υ is an ideal of X. □

Corollary 3.8. Let υ be a pseudo-valuation on a BCI-algebra X. If X is finite order or $X = B(X)$, then I_υ is a closed ideal of X.

Proof. It follows from Theorem 3.7 and Proposition 2.7. □

Remark 3.9. The ideal induced by a pseudo-valuation υ on a BCI-algebra X may not be closed. Consider Example 3.2 part (ii). If $\upsilon(x) = x$, for all $x \in Z$, then I_υ is the set of negative integer which is not a closed ideal of Z.

Theorem 3.10. Let I be an ideal of a BCI-algebra X. Then $I_{\upsilon_I} = I$.

Proof. We have $I_{\upsilon_I} = \{x \in X : \upsilon_I(x) \leq 0\} = \{x \in X : x \in I\} = I$. □

Remark 3.11. The above Theorems do not furnish a one to one correspondence between ideals and pseudo-valuations, because two distinct pseudo-valuations of a given BCI-algebra may induce the same ideal. Consider the following example:

Example 3.12. Let X be a BCI-algebra with the universe $\{0, 1, 2, a, b\}$ such that the operation $*$ is defined by the table below:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

Define $\upsilon_1(0) = \upsilon_1(1) = 0$, $\upsilon_1(2) = 4$, $\upsilon_1(a) = 3$, $\upsilon_1(b) = 5$ and $\upsilon_2(0) = \upsilon_2(1) = 0$, $\upsilon_2(2) = 4$, $\upsilon_2(a) = 2$, $\upsilon_2(b) = 3$. Then υ_1 and υ_2 are two pseudo-valuations on X such that $I_{\upsilon_1} = \{0, 1\} = I_{\upsilon_2}$.

Theorem 3.13. Let υ be a pseudo-valuation on a BCI-algebra X. Define $d_\upsilon : X \times X \rightarrow \Re$ by

$$d_\upsilon(x, y) = \upsilon(x * y) + \upsilon(y * x),$$

for $(x, y) \in X \times X$. Then d_υ is a pseudo-metric on X which is called the *pseudo-metric induced by pseudo-valuation* υ.

Proof. See Theorem 3.6 in [4]. □

Theorem 3.14. Let υ be a pseudo-valuation on a BCI-algebra X such that I_υ is a closed ideal of X. If d_υ is a metric on X, then υ is a valuation.

Proof. Suppose that υ is not a valuation on X. Then there exists $x \in X$ such that $x \neq 0$ and $\upsilon(x) = 0$. Hence $0, x \in I_\upsilon$. Since I_υ is a closed ideal of X, then $0 * x \in I_\upsilon$, that is $\upsilon(0 * x) \leq 0$. We have

$$0 = \upsilon(0) \leq \upsilon(0 * x) + \upsilon(x) = \upsilon(0 * x) \leq 0.$$

Hence $\upsilon(0 * x) = 0$. We get that $d_\upsilon(x, 0) = \upsilon(x * 0) + \upsilon(0 * x) = 0$. Since d_υ is a metric on X, then $x = 0$ which is a contradiction. □

If I_υ is not a closed ideal of X, then the above theorem may not be true. See the following example:

Example 3.15. Consider the set Z of integer, together with the binary operation $*$ defined by $x * y = x - y$. Let $a > o$ be an arbitrary element of Z. Define $\upsilon_a(x) = a - x$, where $x \in Z - \{0\}$ and $\upsilon_a(0) = 0$. Then υ_a is a pseudo-valuation on a BCI-algebra Z, d_υ is a metric space and $I_\upsilon = \{x \in X : a \leq x\} \cup \{0\}$ is not a closed ideal of Z. Since $\upsilon_a(a) = 0$, then υ_a is not a valuation.

Theorem 3.16. Let υ be a valuation on a BCI-algebra X such that $I_\upsilon = \{0\}$. Then d_υ is a metric on X.

Proof. Since $I_\upsilon = \{0\}$, then $\upsilon(x) \geq 0$ for all $x \in X$. Hence d_υ is a metric on X by Theorem 3.20 in [4]. □

If $I_\upsilon \neq \{0\}$, then the above theorem may not be true. Consider υ in Remark 3.9. Then $I_\upsilon \neq \{0\}$ and $d_\upsilon(0, 1) = 0$. Hence d_υ is not a metric on X.

Corollary 3.17. Let υ be a pseudo-valuation on a BCK-algebra X. Then υ is a valuation if and only if d_υ is a metric on X.

Proof. Since υ is a valuation and X is a BCK-algebra, then $I_\upsilon = \{0\}$. By Theorem 3.16, d_υ is a metric on X. Converse follows from Theorem 3.14 and Proposition 2.7. □

Lemma 3.18. Let υ be pseudo-valuation on a BCI-algebra X. Then
(1) $d_\upsilon(x * z, y * z) \leq d_\upsilon(x, y)$,
(2) $d_\upsilon(z * x, z * y) \leq d_\upsilon(x, y)$,
(3) $d_\upsilon(x * y, z * w) \leq d_\upsilon(x * y, z * y) + d_\upsilon(z * y, z * w)$,
for all $x, y, z, w \in X$.

Proof. See Proposition 3.17 in [4]. □

4. Quotient BCI-algebras induced by pseudo valuations

Definition 4.1. Let υ be a pseudo-valuation on a BCI-algebra X. Define the relation θ_υ by:

$$(x, y) \in \theta_\upsilon \qquad \text{if and only if} \qquad d_\upsilon(x, y) = 0,$$

for all $x, y \in X$.

Proposition 4.2. Let υ be a pseudo-valuation on a BCI-algebra X. Then θ_υ is a congruence relation on X which is called the *congruence relation induced* by υ.

Proof. Since θ_υ induced by a pseudo-metric, it is an equivalence relation on X. Suppose that $(x, y), (z, w) \in \theta_\upsilon$. Then we have $d_\upsilon(x, y) = d_\upsilon(z, w) = 0$. By

Lemma 3.18 part (1), we have $d_\upsilon(x*z, y*z) \leq d_\upsilon(x,y) = 0$. By Theorem 3.3 part (3), we obtain that $0 \leq \upsilon((x*z)*(y*z)) + \upsilon((y*z)*(x*z)) = d_\upsilon(x*z, y*z)$. Hence $d_\upsilon(x*z, y*z) = 0$ and then $(x*z, y*z) \in \theta_\upsilon$. Similar proof gives $(y*z, y*w) \in \theta_\upsilon$. Since θ_υ is transitive, then $(x*z, y*w) \in \theta_\upsilon$. Hence θ_υ is a congruence relation on X. □

Definition 4.3. Let υ be a pseudo-valuation on a BCI-algebra X and θ_υ be the congruence relation induced by υ. The set of all equivalence classes $[x]_\upsilon = \{y \in A : (x,y) \in \theta_\upsilon\}$ is denoted by X/υ. On this set, we define $[x]_\upsilon * [y]_\upsilon = [x*y]_\upsilon$. The resulting algebra is denoted by X/υ and is called t*he quotient algebra of X induced by pseudo-valuation υ.*

Theorem 4.4. Let υ be a pseudo-valuation on a BCI-algebra X. Then $(X/\upsilon, *, [0]_\upsilon)$ is a BCI-algebra and $d^*([x]_\upsilon, [y]_\upsilon) = d(x,y)$ is a metric on X/υ. Moreover, the quotient topology on X/υ coincide with the metric topology induced by d^*.

Proof. Since θ_υ is a congruence relation, the operation $*$ is well defined. The proof of (BCI 1) and (BCI 2) is obvious. We only prove (BCI 3). Suppose that $[x]_\upsilon * [y]_\upsilon = [0]_\upsilon$ and $[y]_\upsilon * [x]_\upsilon = [0]_\upsilon$ for some $x, y \in X$. Then $[x*y]_\upsilon = [0]_\upsilon$ and $[y*x]_\upsilon = [0]_\upsilon$ by Definition 4.3. So $(x*y, 0), (y*x, 0) \in \theta_\upsilon$. By definition of θ_υ, the following hold

$$\upsilon(x*y) + \upsilon(0*(x*y)) = 0 \qquad \text{and} \qquad \upsilon(y*x) + \upsilon(0*(y*x)) = 0.$$

By Theorem 2.3 part (9), we have $(0*x)*(0*y) = 0*(x*y)$ and $(0*y)*(0*x) = 0*(y*x)$. Since υ is a pseudo-valuation and order preserving, we obtain that

$$\begin{aligned} \upsilon(0*x) - \upsilon(0*y) &\leq \upsilon((0*x)*(0*y)) = \upsilon(0*(x*y)), \\ \upsilon(0*y) - \upsilon(0*x) &\leq \upsilon((0*y)*(0*x)) = \upsilon(0*(y*x)). \end{aligned}$$

We get that

$$\begin{aligned} \upsilon(0*x) - \upsilon(0*y) + \upsilon(x*y) &\leq \upsilon(0*(x*y)) + \upsilon(x*y) = 0, \\ \upsilon(0*y) - \upsilon(0*x) + \upsilon(y*x) &\leq \upsilon(0*(y*x)) + \upsilon(y*x) = 0. \end{aligned}$$

Therefore $\upsilon(x*y) + \upsilon(y*x) \leq 0$. By Theorem 3.3 part (3), $\upsilon(x*y) + \upsilon(y*x) = 0$. It follows that $(x,y) \in \theta_\upsilon$, that is $[x]_\upsilon = [y]_\upsilon$. Hence $(X/\upsilon, *, [0]_\upsilon)$ is a BCI-algebra. □

Proposition 4.5. Let υ be a pseudo-valuation on a BCI-algebra X such that I_υ is a closed ideal of X. Then $I_\upsilon \subseteq [0]_\upsilon$.

Proof. Let $x \in I_\upsilon$. Then $\upsilon(x) \le 0$. Since I_υ is a closed ideal of X, then $0 * x \in I_\upsilon$. By definition I_υ, $\upsilon(0 * x) \le 0$. We get that $\upsilon(0 * x) + \upsilon(x) \le 0$. By Theorem 3.3 part (3), $\upsilon(0 * x) + \upsilon(x) = 0$. Hence $x \in [0]_\upsilon$. □

If I_υ is a not a closed ideal of X, then the above theorem may not be true in general. For example, we have $I_\upsilon \nsubseteq [0]_\upsilon$ in Remark 3.9.

Proposition 4.6. Let υ be a pseudo-valuation on a BCI-algebra X such that $\upsilon(x) \ge 0$ for all $x \in X$. Then $[0]_\upsilon \subseteq I_\upsilon$.

Proof. Let $x \in [0]_\upsilon$. Then $(0, x) \in \theta_\upsilon$. By definition θ_υ, we have $\upsilon(0*x)+\upsilon(x) = 0$. Since $\upsilon(x) \ge 0$ for all $x \in X$, we obtain $\upsilon(0 * x) = \upsilon(x) = 0$. Hence $x \in I_\upsilon$ by definition I_υ. □

If we do not have $\upsilon(x) \ge 0$ for all $x \in X$, then the above theorem may not be true. Consider Example 3.15, we have $I_\upsilon \nsubseteq [0]_\upsilon$.

Corollary 4.7. Let υ be a pseudo-valuation on a BCI-algebra X such that $\upsilon(x) \ge 0$ for all $x \in X$ and I_υ is a closed ideal of X. Then $I_\upsilon = [0]_\upsilon$.

Proof. It follows from Proposition 4.5 and Proposition 4.6. □

Proposition 4.8. Let υ be a pseudo-valuation on a BCI-algebra X and I_υ be the ideal induced by υ. Then $\theta_{I_\upsilon} \subseteq \theta_\upsilon$.

Proof. Let $(x, y) \in \theta_{I_\upsilon}$. Then $x * y, y * x \in I_\upsilon$. We have $\upsilon(x * y) \le 0$ and $\upsilon(y * x) \le 0$, by definition I_υ. Thus $\upsilon(x * y) + \upsilon(y * x) \le 0$. By Theorem 3.3 part (3), $\upsilon(x*y)+\upsilon(y*x) = 0$. It follows that $(x, y) \in \theta_\upsilon$. Hence $\theta_{I_\upsilon} \subseteq \theta_\upsilon$. □

In the above theorem, the opposite inclusion may not hold. See Example 3.2 part (2).

Proposition 4.9. Let υ be a pseudo-valuation on a BCI-algebra X such that $\upsilon(x) \ge 0$ for all $x \in X$ and I_υ be the ideal induced by υ. Then $\theta_\upsilon \subseteq \theta_{I_\upsilon}$.

Proof. Let $(x, y) \in \theta_\upsilon$. Then $\upsilon(x * y) + \upsilon(y * x) = 0$. Since $\upsilon(x) \ge 0$ for all $x \in X$, we obtain that $\upsilon(x * y) = 0$ and $\upsilon(y * x) = 0$. By definition I_υ, we get that $x * y, y * x \in I_\upsilon$. It follows that $(x, y) \in \theta_\upsilon$. Hence $\theta_\upsilon \subseteq \theta_{I_\upsilon}$. □

Proposition 4.10. Let I be an ideal of a BCI-algebra X. Then $\theta_I = \theta_{\upsilon_I}$.

Proof. Let $(x,y) \in \theta_I$. Then $x * y, y * x \in I$ by Theorem 2.10. We have $\upsilon_I(x * y) = \upsilon_I(y * x) = 0$, by Theorem 3.6. Hence $d_\upsilon(x,y) = 0$ and then $(x,y) \in \theta_{\upsilon_I}$.
Conversely, let $(x,y) \in \theta_{\upsilon_I}$. Then $\upsilon_I(x * y) + \upsilon_I(y * x) = 0$. Since $\upsilon_I(x) \geq 0$ for all $x \in X$, we obtain that $\upsilon_I(x * y) = \upsilon_I(y * x) = 0$, that is $x * y, y * x \in I$. Hence $(x,y) \in \theta_I$. □

Theorem 4.11. Let υ_1 and υ_2 be two different pseudo-valuations on a BCI-algebra X such that $[0]_{\upsilon_1} = [0]_{\upsilon_2}$. Then θ_{υ_1} and θ_{υ_2} coincide, thus $X/\upsilon_1 = X/\upsilon_2$.

Proof. Let $(x,y) \in \theta_{\upsilon_1}$. Then $(x * y, 0) = (x * y, y * y) \in \theta_{\upsilon_1}$. It follows that $x * y \in [0]_{\upsilon_1}$. Similarly, we can show that $y * x \in [0]_{\upsilon_1}$. By assumption $[0]_{\upsilon_1} = [0]_{\upsilon_2}$, so we get that

$$[x]_{\upsilon_2} * [y]_{\upsilon_2} = [x * y]_{\upsilon_2} = [0]_{\upsilon_2} \qquad \text{and} \qquad [y]_{\upsilon_2} * [x]_{\upsilon_2} = [y * x]_{\upsilon_2} = [0]_{\upsilon_2}$$

Since X/υ_2 is a BCI-algebra, then $[x]_{\upsilon_2} = [y]_{\upsilon_2}$. Hence $(x,y) \in \theta_{\upsilon_2}$ and then $X/\upsilon_1 = X/\upsilon_2$. It follows that $X/\upsilon_2 = X/\upsilon_1$. □

Lemma 4.12. Let υ be a pseud-valuation on a BCI-algebra X and I be an ideal of X such that $[0]_\upsilon \subseteq I$. Denote $I/\upsilon = \{[x]_\upsilon : x \in I\}$. Then
(1) $x \in I$ if and only if $[x]_\upsilon \in I/\upsilon$ for any $x \in X$,
(2) I/υ is an ideal of X/υ.

Proof. (1) Suppose that $[x]_\upsilon \in I/\upsilon$. Then there exists $y \in I$ such that $[x]_\upsilon = [y]_\upsilon$. Hence $(x,y) \in \theta_\upsilon$. It follows that $(x * y, 0) \in \theta_\upsilon$. We get that $x * y \in [0]_\upsilon$. Since $[0]_\upsilon \subseteq I$, we have $x * y, y \in I$. Hence $x \in I$. The converse is trivial.
(2) Since $0 \in I$, then $[0]_\upsilon \in I/\upsilon$ by part (1). Let $[x]_\upsilon * [y]_\upsilon, [y]_\upsilon \in I/\upsilon$. By Definition 4.3, $[x]_\upsilon * [y]_\upsilon = [x * y]_\upsilon$. We have $x * y, y \in I$ by part (1). Since I is an ideal, $x \in I$. We get that $[x]_\upsilon \in I/\upsilon$. Therefore I/υ is an ideal of X/υ. □

Lemma 4.13. Let υ be a pseudo-valuation on a BCI-algebra X and J be an ideal of X/υ. Then $I = \{x \in X : [x]_\upsilon \in J\}$ is an ideal of X such that $[0]_\upsilon \subseteq I$.

Proof. It is clear that $0 \in [0]_\upsilon \subseteq I$. Suppose that $x * y, y \in I$. Then $[y]_\upsilon, [x * y]_\upsilon = [x]_\upsilon * [y]_\upsilon \in J$. Since J is an ideal of X/υ, then $[x]_\upsilon \in J$. By definition I, we obtain $x \in I$. Hence I is an ideal of X. □

Theorem 4.14. Let υ be a pseudo-valuation on a BCI-algebra X, $I(X,\upsilon)$ the collection of all ideals of X containing $[0]_\upsilon$, and $I(X/\upsilon)$ the collection of all ideals of X/υ. Then $\varphi : I(X,\upsilon) \to I(X/\upsilon)$, $I \to I/\upsilon$, is a bijection.

Proof. It follows from Lemma 4.12 and Lemma 4.13. □

Lemma 4.15. Let X and Y be BCI-algebras, $f : X \to Y$ a homomorphism and υ a pseudo-valuation on Y. Then $\upsilon \circ f : X \to \Re$ defined by $\upsilon \circ f(x) = \upsilon(f(x))$ for all $x \in X$ is a pseudo-valuation on X.

Proof. The proof is straightforward. □

Theorem 4.16. Let X and Y be BCI-algebras, $f : X \to Y$ an epimorphism and υ a pseudo-valuation on Y. Then $X/\upsilon \circ f \cong Y/\upsilon$.

Proof. By Lemma 4.15 and Theorem 4.4, we have $X/\upsilon \circ f$ and Y/υ are BCI-algebras. Define $\psi : X/\upsilon \circ f \to Y/\upsilon$ by $\psi([x]_{\upsilon\circ f}) = [f(x)]_\upsilon$ for all $x \in X$.
(1) Suppose that $[x]_{\upsilon\circ f} = [y]_{\upsilon\circ f}$. Then $(\upsilon \circ f)(x * y) + (\upsilon \circ f)(y * x) = 0$. Since f is a homomorphism, then $\upsilon(f(x) * f(y)) + \upsilon(f(y) * f(x)) = 0$. We obtain that $[f(x)]_\upsilon = [f(y)]_\upsilon$. We get that $\psi([x]_{\upsilon\circ f}) = \psi([y]_{\upsilon\circ f})$, that is ψ is well define.
(2) We show that ψ is a homomorphism. Since f is a homomorphism,
(i) $\psi([0]_{\upsilon\circ f}) = [f(0)]_\upsilon = [0]_\upsilon$,
(ii) $\psi([x]_{\upsilon\circ f} * [y]_{\upsilon\circ f}) = \psi([x * y]_{\upsilon\circ f}) = [f(x * y)]_\upsilon = [f(x) * f(y)]_\upsilon = [f(x)]_\upsilon * [f(y)]_\upsilon = \psi([x]_{\upsilon\circ f}) * \psi([y]_{\upsilon\circ f})$.
(3) Let $[y]_\upsilon \in Y/\upsilon$ be arbitrary. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Hence $\psi([x]_{\upsilon\circ f}) = [f(x)]_\upsilon = [y]_\upsilon$ and ψ is surjective.
(4) We prove that ψ is one to one. Suppose that $\psi([x]_{\upsilon\circ f}) = \psi([y]_{\upsilon\circ f})$. Then $[f(x)]_\upsilon = [f(y)]_\upsilon$. We get that $\upsilon(f(x) * f(y)) + \upsilon(f(y) * f(x)) = 0$. Since f is a homomorphism, then $(\upsilon \circ f)(x * y) + (\upsilon \circ f)(y * x) = 0$. We obtain $[x]_{\upsilon\circ f} = [y]_{\upsilon\circ f}$. Hence $X/\upsilon \circ f \cong Y/\upsilon$. □

Lemma 4.17. Let υ be a pseudo-valuation on a BCI-algebra X and X/υ be the corresponding quotient algebra. Then the map $\pi : X \to X/\upsilon$ defined by $\pi(x) = [x]_\upsilon$ for all $x \in X$ is an epimorphism.

Corollary 4.18. Let υ be a pseudo-valuation on a BCI-algebra X and X/υ the corresponding quotient algebra. For each pseudo-valuation $\bar{\upsilon_1}$ on a BCI-algebra X/υ, there exists a pseudo-valuation υ_1 on a BCI-algebra X, such that $\upsilon_1 = \bar{\upsilon_1} \circ \pi$.

Proof. It follows from Lemma 4.15 and Lemma 4.17. □

Theorem 4.19. Let υ be a pseudo-valuation on a BCI-algebra X such that $\upsilon(x) \geq 0$ for all $x \in X$. Then $\overline{\upsilon} : X/\upsilon \to \Re$ define by $\overline{\upsilon}([x]_\upsilon) = \upsilon(x)$ is a pseudo-valuation on X/υ.

Proof. It is enough to show that $\overline{\upsilon}$ is well defined. Let $[x]_\upsilon = [y]_\upsilon$. Since $\upsilon(x) \geq 0$, then $\upsilon(x*y) = \upsilon(y*x) = 0$. We have $x*(x*y) \leq y$. By Theorem 3.3 part (1), $\upsilon(x*(x*y)) \leq \upsilon(y)$. It follows that $\upsilon(x*(x*y)) + \upsilon(x*y) \leq \upsilon(y) + \upsilon(x*y)$. Therefore $\upsilon(x) \leq \upsilon(x*(x*y)) + \upsilon(x*y) \leq \upsilon(y)$. Similarly, we can show that $\upsilon(y) \leq \upsilon(x)$. Therefore $\upsilon(y) = \upsilon(x)$ and then $\overline{\upsilon}$ is well defined. □

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