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# On the WZ Factorization of the Real and Integer Matrices

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Abstract. The *QIF* (Quadrant Interlocking Factorization) method of Evans and Hatzopoulos solves linear equation systems using *WZ* factorization. The WZ factorization can be faster than the *LU* factorization because, it performs the simultaneous evaluation of two columns or two rows. Here, we present a method for computing the real and integer *WZ* and *ZW* factorizations by using the null space generators of some special nested submatrices of a matrix $A$.

**Keywords:** Linear systems, Quadrant interlocking factorization, *WZ* factorization, *ZW* factorization, Null space generator.

**2000 Mathematics subject classification:** 15A03, 15A21, 15A23, 34C20, 49M27.

## 1. Introduction

Linear systems arise frequently in scientific and engineering computing. Various serial and parallel algorithms have been introduced for their serial solution [9, 4]. The *QIF* (Quadrant Interlocking Factorization) algorithm, introduced by Evans and Hatzopoulos, is a numerical method for finding a solution for systems of the type $Ax = b$, where $A$ is a nonsingular matrix of dimensions

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$n \times n$, $x$ is an unknown column vector, and $b$ is the independent term vector provided. The $QIF$ method is based on the $WZ$ factorization of the coefficient matrix $A$. The main advantage of this factorization is that it presents a complexity order less than of the $LU$ decomposition due to the fact that it performs the simultaneous evaluation of two columns or two rows. A detailed description of this algorithm for real and complex matrices can be found in [4, 5, 10, 11]. Golpar-Raboky and Mahdavi-Amiri presented new algorithms for computing the real and integer $WZ$ and $ZW$ matrix factorizations using $ABS$ algorithms and the extended rank reduction process [6, 7, 8, 16]. Recently, some authors have considered simultaneous matrix decompositions [12, 13, 14].

The $WZ$ factorization is used for solving Markovian linear systems [2] and network modeling [3], preconditioning of sparse matrices [18] and eigenvalue problems [15].

Let $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ stand for the real number, and the set of all $m \times n$ matrices over $\mathbb{R}$ and $A^T$ denotes the transpose of $A$. Let $A = (a_1, \cdots, a_m)^T \in \mathbb{R}^{m\times n}$. Assume that $a_{k_1}^T, \cdots, a_{k_i}^T$ be the rows of $A$ and $H_1 \in \mathbb{R}^{n\times n}$ be an arbitrary nonsingular matrix. For $j = 1, \cdots, i$ update $H_j$ by

$$H_{j+1} = H_j - \frac{H_j a_{k_j} w_j^T H_j}{w_j^T H_j a_{k_j}}, \tag{1.1}$$

where $w_j \in \mathbb{R}^n$ such that $w_j^T H_j a_{k_j} \neq 0$. Then, we have

$$a_{k_i}^T H_{j+1}^T = 0,\ i = 1, \cdots, j, \tag{1.2}$$

and the linear combination of the columns of $H_{i+1}^T$ generates the null space of $\{a_{k_1}, \cdots a_{k_i}\}$ (see [1]).

Matrices $H_i$ are generalizations of (oblique) projection matrices. They probably first appeared in a book by Wedderburn [19]. They have been named ***Abbafians*** since the First International Conference on ABS methods(Luoyang, China, 1991) and this name will be used here.

**Notation:** Let $A \in \mathbb{R}^{n\times n}$. Here and subsequently $J_n = \{j_1, \cdots, j_n\}$ denotes a permutation of $\mathcal{I}_n = \{1, 2, \cdots, n\}$ and, for $k = 1, \cdots, n$, $J_k = \{j_1, \cdots, j_k\}$ denotes a subset of $J_n$. Let

$$A_{J_k} = (a_{i,j}),\ i, j \in J_k. \tag{1.3}$$

denotes a submatrix of $A$, and

$$J_1 \subset J_2 \subset \cdots \subset J_n, \tag{1.4}$$



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and $\{A_{J_k}\}_{k=1}^n$ be a sequence of nested submatrices of $A$. The following theorem describes a necessary and sufficient condition for nonsingularity $A_{J_k},\ k = 1, \cdots, n$.

**Theorem 1.1.** *(Nested submatrices) Let $A \in \mathbb{R}^{n\times n}$ and $H_1 = I$. Then the nested submatrices $A_{J_i},\ i = 1, \cdots, n$, are nonsingular if and only if $e_{j_i}^T H_i a_{j_i} \neq 0,\ i = 1, \cdots, n$.*

*Proof.* Follow the lines of the proof for Theorem 6.5 in [1] by replacing $i$ to $j_i$. □

From (1.1) and Theorem 1.1 we have the following result.

**Theorem 1.2.** *Let $A \in \mathbb{R}^{n\times n}$, $H_1 = I$, and for $i = 1, \cdots, n$, $e_{j_i}^T A H_i e_{j_i} \neq 0$. Then,*

$$H_{i+1} = H_i - \frac{H_i a_{j_i} e_{j_i}^T H_i}{e_{j_i}^T H_i a_{j_i}}, \tag{1.5}$$

*is well defined.*

The parameter choices in Theorem 1.2, induce a structure in the matrix $H_i$, described by the following theorem.

**Theorem 1.3.** *Let the conditions of Theorem 1.2 be satisfied and $H_{i+1}$ defined by (1.5). Then, the following properties hold:*

*(a) The $j$th row of $H_{i+1}$ is zero, for $j \in J_i$.*

*(b) The $j$th column of $H_{i+1}$ is equal to the $j$th column of $H_1$, for $j \notin J_i$.*

*Proof.* See Theorem 6.3 in [1]. □

In this paper we present new algorithms for computing the *WZ* and *ZW* factorizations using null space of special submatrices of the matrix $A$.

The structure of this paper is organized as follows. In Section 2, we discuss our proposed algorithm for the *WZ* factorization of a matrix $A$ by using null space of special submatrices of $A$. In Section 3, we propose a new algorithm for computing the $WZ$ and $ZW$ factorizations. In Section 4, we report a numerical experiment. We conclude in Section 5.

## 2. The *WZ* factorization

The *WZ* factorization is a parallel method for solving dense linear systems of the form

$$Ax = b, \tag{2.1}$$

where $A$ is a square $n \times n$ matrix, and $b$ is an $n$-vector.



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**Definition 2.1.** *Let $s$ be a real number and denote by $\lfloor s \rfloor$ ($\lceil s \rceil$), the greatest (least) integer less (greater) than or equal to $s$.*

**Definition 2.2.** *We say that a matrix $A$ is factorized in the form $WZ$ if*

$$A = WZ, \tag{2.2}$$

*where the matrices $W$ and $Z$ have the following structures:*

$$W = \begin{pmatrix} * & 0 & \cdots & 0 & * \\ * & * & 0 & * & * \\ * & * & * & * & * \\ * & * & 0 & * & * \\ * & 0 & \cdots & 0 & * \end{pmatrix}, Z = \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & 0 \\ \vdots & 0 & * & 0 & \vdots \\ 0 & * & * & * & 0 \\ * & * & * & * & * \end{pmatrix} \tag{2.3}$$

where stars stand for possible nonzero entries.

The matrices $W$ and $Z$ have two zero opposite quadrants. Then, we refer to $W$ and $Z$ as the interlocking quadrant factors of $A$. The factorization is unique if $W$ has $1's$ on the main diagonal and $0's$ on the cross diagonal entries(see [17]).

Now, we give a characterization for the existence of the $WZ$ factorization of $A$.

**Theorem 2.1.** *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. $A$ has quadrant interlocking factorization QIF, A=WZ, if and only if for every $k$, $1 \le k \le s$, where $s = \lfloor n/2 \rfloor$ if $n$ is even and $s = \lceil n/2 \rceil$ if $n$ is odd, the $2k \times 2k$ submatrix*

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix} \tag{2.4}$$

*of $A$ is invertible. Moreover, the factorization is unique.*

*Proof.* See Theorem 2 in [17]. □

If $A \in \mathbb{R}^{n\times n}$ is nonsingular, then the $WZ$ factorization with pivoting can always be carried out. Whenever $\Delta_k$ is nonsingular, it is always possible to interchange the rows $k \le i \le (n-k+1)$. These row interchanges can be viewed in a matrix form as premultiplication by a permutation matrix. Thus, we have the following result.

**Theorem 2.2.** *If $A \in \mathbb{R}^{n\times n}$ is nonsingular, then the with pivoting $WZ$ factorization can always be carried out, that is, a row permutation matrix $P$ and the factors $W$ and $Z$ exist so that, $PA = WZ$.*



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*Proof.* See [17]. □

Let $A \in \mathbb{R}^{n\times n}$ and there exists a $WZ$ factorization without pivoting of $A$. Let, $n$ be an even number. Here, we present a new algorithm for computing the $WZ$ factorization of $A$ using null space of the sequence submatrices

$$\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_{n/2}. \tag{2.5}$$

For $k = 1, \cdots, s$, where $s = \dfrac{n}{2}$ consider $\Delta_k$ defined by (2.4). Let, the rows of $H_k$ generate the null space of $\Delta_k$ expect the $k$th and the $(k+1)$th rows. Let $e_j \in \mathbb{R}^{2k}$ be the $j$th unit vector, then we have,

$$e_i^T \Delta_k {H_k}^T = 0,\ i \neq k, k+1, \tag{2.6}$$

and

$$(e_k^T, e_{k+1}^T)\Delta_k {H_k}^T \neq 0. \tag{2.7}$$

Therefore, there exist $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Delta_k^T e_k \neq 0, \alpha_2 = e_{j_2}^T H_k \Delta_k^T e_{k+1} \neq 0. \tag{2.8}$$

Let $\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1 \in \mathbb{R}^{2k}$ and $\mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2 \in \mathbb{R}^{2k}$. Then, we have

$$\Delta_k \mathcal{T}_k = (\underbrace{0...0}_{k-1}, 1, \underbrace{0, ..., 0}_{k})^T,\ \Delta_k \mathcal{Y}_k = (\underbrace{0...0}_{k}, 1, \underbrace{0, ..., 0}_{k-1})^T. \tag{2.9}$$

Now, let

$$\bar{z}_k = (t_1, \cdots, t_k, \underbrace{0, \cdots, 0}_{n-2k}, t_{k+1}, \cdots, t_{2k})^T, \tag{2.10}$$

and

$$\bar{z}_{n-k+1} = (y_1, \cdots, y_k, \underbrace{0, \cdots, 0}_{n-2k}, y_{k+1}, \cdots, y_{2k})^T, \tag{2.11}$$

then, we have

$$w_k = A\bar{z}_k = (\underbrace{0, \cdots, 0}_{k-1}, 1, w_{k+1,k}, \cdots, w_{n-k,k}, \underbrace{0, \cdots, 0}_{k})^T \tag{2.12}$$

and

$$w_{n-k+1} = A\bar{z}_{n-k+1} = (\underbrace{0, \cdots, 0}_{k}, w_{k+1,n-k+1}, \cdots, w_{n-k,n-k+1}, 1, \underbrace{0, \cdots, 0}_{k-1})^T. \tag{2.13}$$



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$$\tilde{Z} = (\bar{z}_1, \cdots, \bar{z}_n),\ W = (w_1, \cdots, w_n), \tag{2.14}$$

then, we have,

$$A\bar{Z} = W \Rightarrow A = WZ,\ Z = \bar{Z}^{-1}.$$

Here, we are ready to present the $WZ$ algorithm. Without loss of generality we assume that $A$ is an even order matrix.

**Algorithm 1. *WZ* algorithm**

(1) Let $A^{(0)} = A$, $k = 1$, $s = n/2$.

(2) Compute $P_k$, $A^{(k)} = P_k A^{(k-1)}$ where, $P_k$ is a permutation matrix and $\Delta_k$ is nonsingular.

(3) Compute $H_k^k$, so that the rows of $H_k^k$ present the null space of the rows of $\Delta_k$ except the $k$th and $(k+1)$th rows.

(4) Determine $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_k \neq 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{k+1} \neq 0. \tag{2.15}$$

(5) Compute,

$$\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1 \text{ and } \mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2.$$

(6) Compute,

$$\bar{z}_k = (t_1, \cdots, t_k, \underbrace{0, \cdots, 0}_{n-2k}, t_{k+1}, \cdots, t_{2k})^T, \tag{2.16}$$

and

$$\bar{z}_{n-k+1} = (y_1, \cdots, y_k, \underbrace{0, \cdots, 0}_{n-2k}, y_{k+1}, \cdots, y_{2k})^T, \tag{2.17}$$

(7) **If** $k < s$ **then** k=k+1 and **go to** (2).

(8) Compute

$$PA = WZ,$$



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where, $P = P_s \cdots P_1,\ \bar{Z} = (\bar{z}_1, \cdots, \bar{z}_n),\ W = PA\bar{Z}$ and $Z = \bar{Z}^{-1}$.

(9) **Stop**.

The integer $WZ$ factorization of an integer matrix, can be calculated as the real case if it exists. Here, we present the conditions for existence of the integer $WZ$ factorization of an integer matrix.

**Definition 2.3.** *$A \in \mathbb{Z}^{n\times n}$ is a unimodular matrix if and only if $|det(A)| = 1$.*

If $A$ is unimodular, then $A^{-1}$ is also unimodular.

**Definition 2.4.** *We say that a matrix $A$ is factorized in an integer WZ form if*

$$A = WZ, \tag{2.18}$$

*where the matrices W and Z are matrices with integer entries defined by (2.3).*

According to Theorem 2.1, we have the following result.

**Theorem 2.3.** *Let $A \in \mathbb{Z}^{n\times n}$ and the submatrices $\Delta_k$ defined by (2.4) be unimodular, then A has an integer WZ factorization.*

For computing an integer $WZ$ factorization (if there exits), in the $k$th step $H_k$ generates the integer null space of $\Delta_k$ expect the $k$th and the $(k+1)$th rows. Furthermore, in (2.8) we choose two integer vectors $j_1$ and $j_2$ such that

$$\alpha_1 = e_{j_1}^T H_k \Delta_k^T e_k = gcd(H_k \Delta_k^T e_k), \alpha_2 = e_{j_2}^T H_k \Delta_k^T e_{k+1} = gcd(H_k \Delta_k^T e_{k+1}), \tag{2.19}$$

where, $gcd(x)$ is the greatest common divisor of entries of $x$.

**Definition 2.5.** *A matrix $A \in \mathbb{Z}^{n\times n}$ is called totally unimodular if each square submatrix of A has determinant equal to $0$, $+1$, or $-1$. In particular, each entry of a totally unimodular matrix is $0$, $+1$, or $-1$.*

**Corollary 2.1.** *Every totally unimodular symmetric positive definite matrix has an integer WZ factorization.*

## 3. The $ZW$ factorization

**Definition 3.1.** *We say that a matrix A is factorized in the form ZW if*

$$A = ZW, \tag{3.1}$$

*where the matrices W and Z are defined as (2.3)*

where the empty bullets stand for zero and the other bullets stand for possible nonzero entries.

The factorization is unique if $Z$ has $1's$ on the main diagonal and $0's$ on the



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cross diagonal.

Without loss of generality, suppose that $n$ be an even number and $s = \frac{n}{2}$. Here, we present a new algorithm for computing the $ZW$ factorization of $A$ using null space of the sequence of submatrices

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{n/2} \tag{3.2}$$

where

$$\Lambda_k = \begin{pmatrix} a_{s-k+1,s-k+1} & \cdots & a_{s-k+1,s+k} \\ \vdots & \cdots & \vdots \\ a_{s+k,s-k+1} & \cdots & a_{s+k,s+k} \end{pmatrix}_{2k,2k} . \tag{3.3}$$

**Theorem 3.1.** *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. A has a ZW factorization, if and only if for every $k$, $1 \le k \le n/2$, the submatrix $\Lambda_k$ defined by (3.3) be invertible.*

*Proof.* The proof follows the lines of the proof for Theorem 2 in [17] replacing $\Delta_i$ by $\Lambda_i$. □

Let $\Lambda_k$ be nonsingular, for $k = 1, \cdots, n/2$. Let, the rows of $H_k$ generates the null space of $\Lambda_k$ expect the first and the last rows. Let $e_i \in \mathbb{R}^{2k}$ be the $i$th unit vector (i.e. the $i$th element is 1, otherwise 0). Then, we have,

$$e_i^T \Lambda_k {H_k}^T = 0,\ i \ne 1, 2k,\ (e_1^T, e_{2k}^T)\Lambda_k {H_k}^T \ne 0, \tag{3.4}$$

then there exists $1 \le j_1, j_2 \le 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 \ne 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} \ne 0. \tag{3.5}$$

Let $\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1$ and $\mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2$. Then, we have

$$\Lambda_k t = (1, \underbrace{0...0}_{2k-1})^T,\ \Lambda_k y = (\underbrace{0...0}_{2k-1}, 1)^T. \tag{3.6}$$

Now, let

$$\bar{w}_{\frac{n}{2}-k+1} = (\underbrace{0, \cdots, 0}_{(n-2k)/2}, t_1, \cdots, t_{2k}, \underbrace{0, \cdots, 0}_{(n-2k)/2})^T, \tag{3.7}$$

and

$$\bar{w}_{\frac{n}{2}+k} = (\underbrace{0, \cdots, 0}_{(n-2k)/2}, y_1, \cdots, y_{2k}, \underbrace{0, \cdots, 0}_{(n-2k)/2})^T, \tag{3.8}$$



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then, we have

$$z_{\frac{n}{2}-k+1} = A\bar{w}_{\frac{n}{2}-k+1} = \\ (z_{1,\frac{n}{2}-k+1}, \cdots, z_{\frac{(n-2k)}{2},\frac{n}{2}-k+1}, 1, \underbrace{0, \cdots, 0}_{2k-1}, z_{\frac{(n+2k)}{2}+1,\frac{n}{2}-k+1}, \cdots, z_{n,\frac{n}{2}-k+1})^T \tag{3.9}$$

and

$$z_{\frac{n}{2}+k} = A\bar{w}_{\frac{n}{2}+k} = \\ (z_{1,\frac{n}{2}+k}, \cdots, z_{\frac{(n-2k)}{2},\frac{n}{2}+k}, \underbrace{0, \cdots, 0}_{2k-1}, 1, z_{\frac{(n+2k)}{2}+1,\frac{n}{2}+k}, \cdots, z_{n,\frac{n}{2}+k})^T. \tag{3.10}$$

$$\bar{W} = (\bar{w}_1, \cdots, \bar{w}_n),\ Z = (z_1, \cdots, z_n),$$

then, we have,

$$A\bar{W} = Z \Rightarrow A = ZW,\ W = \bar{W}^{-1}$$

Here, we are ready to present the $ZW$ algorithm. Without loss of generality we assume that $A$ is an even order matrix.

**Algorithm 2. *ZW* algorithm**

(1) Let $A^{(0)} = A$, $k = 1$, $s = n/2$.

(2) Compute $P_k$, $A^{(k)} = P_k A^{(k-1)}$ where, $P_k$ is a permutation matrix and $\Lambda_k$ is nonsingular.

(3) Let the rows of $H_k$ generate the null space of $\Lambda_k$ expect the first and the last rows.

(4) Determine $1 \leq j_1, j_2 \leq 2k$ such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 \neq 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} \neq 0. \tag{3.11}$$

(5) Compute,

$$\mathcal{T}_k = (t_1, \cdots, t_{2k}) = {H_k}^T e_{j_1}/\alpha_1 \text{ and } \mathcal{Y}_k = (y_1, \cdots, y_{2k}) = {H_k}^T e_{j_2}/\alpha_2.$$

(6) Compute,

$$\bar{w}_{\frac{n}{2}-k+1} = (\underbrace{0, \cdots, 0}_{(n-2k)/2}, t_1, \cdots, t_{2k}, \underbrace{0, \cdots, 0}_{(n-2k)/2})^T, \tag{3.12}$$



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and

$$\bar{w}_{\frac{n}{2}+k} = (\underbrace{0,\cdots,0}_{(n-2k)/2}, y_1, \cdots, y_{2k}, \underbrace{0,\cdots,0}_{(n-2k)/2})^T, \tag{3.13}$$

(7) **If** $k < s$ **then** k=k+1 and **go to** (2).

(8) Compute

$$PA = ZW,$$

where, $P = P_s \cdots P_1$, $\bar{W} = (\bar{w}_1, \cdots, \bar{w}_n)$, $Z = PA\bar{W}$ and $W = \bar{W}^{-1}$.

(9) **Stop**.

We can also calculate the integer $ZW$ factorization of an integer matrix $A$. The existence conditions are the same as Theorem 2.3 by replacing $\Delta$ by $\Lambda$.

**Theorem 3.2.** *Let $A \in \mathbb{Z}^{n\times n}$ and the submatrices $\Lambda_k$ be unimodular, then $A$ has an integer $ZW$ factorization.*

For computing an integer $ZW$ factorization (if there exits), in the $k$th step $H_k$ generates the integer null space of $\Lambda_k$ expect the first and the last rows. Furthermore, in (3.5) we choose two integer vectors $j_1$ and $j_2$ such that

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 = gcd(H_k \Lambda_k^T e_1), \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} = gcd(H_k \Lambda_k^T e_{2k}). \tag{3.14}$$

**Corollary 3.1.** *Every totally unimodular symmetric positive definite matrix has an integer $ZW$ factorization.*

## 4. Examples

In this section, we present some numerical illustrations of our proposed algorithms to compute the $WZ$ and $ZW$ factorizations of real and integer matrices.

Example 4.1. Consider the following matrix,

$$A = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$



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Upon an application of Algorithm 1 for computing the $WZ$ factorization, we obtain the following results:

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7895 & 1 & 0 & 0.0526 \\ 0.1053 & 0 & 1 & 0.4737 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; Z = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 0 & 1.7895 & 0.1053 & 0 \\ 0 & 0.1053 & 2.9474 & 0 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

Example 4.2. Consider the following matrix

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 & 2.5 & 2.5 \\ 3 & 3 & 3.5 & 2.5 & 3 & 2.5 \\ 1.5 & 3.5 & 1 & 2.5 & 2 & 2.5 \\ 2 & 2.5 & 2.5 & 4 & 1.5 & 3 \\ 2.5 & 3 & 2 & 1.5 & 2 & 2.5 \\ 2.5 & 2.5 & 2.5 & 3 & 2.5 & 1 \end{pmatrix}.$$

By applying Algorithm 2 for computing the $ZW$ factorization we have

$$Z = \begin{pmatrix} 1 & 0.3219 & 1.5 & 0.5 & 0.6452 & -0.8439 \\ 0 & 1 & 3.5 & 0.6250 & 1.6613 & 0 \\ 0 & 0 & 1 & 0.6250 & 0 & 0 \\ 0 & 0 & 2.5 & 1 & 0 & 0 \\ 0 & 0.7055 & 2 & 0.3750 & 1 & 0 \\ -1.5774 & 0.3425 & 2.5000 & 0.7500 & 0.7419 & 1.0000 \end{pmatrix},$$

and

$$W = \begin{pmatrix} -0.4855 & 0 & 0 & 0 & 0 & 0 \\ 6.2826 & 8.1111 & 0 & 0 & 0 & 8.5331 \\ -0.4444 & -3.4444 & 1 & 0 & -1.8889 & -1.1111 \\ 3.1111 & 11.1111 & 0 & 4 & 6.2222 & 5.7778 \\ -2.2100 & 0 & 0 & 0 & 3.4444 & -3.4644 \\ 0 & 0 & 0 & 0 & 0 & -0.9075 \end{pmatrix}.$$

Example 4.3. Consider the following integer real matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 2 & 0 & 3 & 1 & 1 \\ -1 & 0 & 5 & -1 & 7 & 2 \\ 1 & 3 & -1 & 8 & 2 & 1 \\ -1 & 1 & 7 & 2 & 15 & 4 \\ -1 & 1 & 2 & 1 & 4 & 2 \end{pmatrix}.$$

Upon an application of Algorithm 1 for computing the integer $WZ$ factorization, we obtain the following results:



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$$W=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&1\\-1&-1&1&0&1&1\\1&1&0&1&-1&2\\-1&-2&0&0&1&3\\-1&0&0&0&0&1\end{pmatrix},\ Z=\begin{pmatrix}1&0&-1&1&-1&-1\\0&1&-1&1&-2&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&1&-1&1&0\\0&1&1&2&3&1\end{pmatrix}.$$

Example 4.4. Consider the following matrix

$$A=\begin{pmatrix}5&4&-1&1&-3&-2\\4&7&-1&1&-4&0\\-1&-1&1&0&1&1\\1&1&0&1&-1&0\\-3&-4&1&-1&3&1\\-2&0&1&0&1&3\end{pmatrix}.$$

By applying Algorithm 2 for computing the integer $ZW$ factorization we have

$$Z=\begin{pmatrix}1&0&-1&1&-1&-1\\0&1&-1&1&-2&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&1&-1&1&0\\0&1&1&0&0&1\end{pmatrix},\ W=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&1\\-1&-1&1&0&1&1\\1&1&0&1&-1&0\\-1&-2&0&0&1&0\\-1&0&0&0&0&1\end{pmatrix}.$$

## 5. Conclusion

Parallel implicit matrix elimination schemes for the solution of linear systems were introduced by Evans. In this paper we showed how to compute the real (integer) $WZ$ and $ZW$ factorizations by using the null space generators of particular submatrices of a given matrix $A$.

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