A Functional Characterization of the Hurewicz Property

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Abstract. For a Tychonoff space $X$, we denote by $C_p(X)$ the space of all real-valued continuous functions on $X$ with the topology of pointwise convergence. We study a functional characterization of the covering property of Hurewicz.

Keywords: $U_{fin}(O, \Omega)$, Hurewicz property, Selection principles, $C_p$ theory, $U_{fin}(O, \Gamma)$.


1. Introduction

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $A$ and $B$ be sets consisting of families of subsets of an infinite set $X$. Then:

$S_1(A, B)$ is the selection hypothesis: for each sequence $\langle A_n : n \in \mathbb{N} \rangle$ of elements of $A$ there is a sequence $\langle b_n : n \in \mathbb{N} \rangle$ such that for each $n$, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $B$.

$S_{fin}(A, B)$ is the selection hypothesis: for each sequence $\langle A_n : n \in \mathbb{N} \rangle$ of elements of $A$ there is a sequence $\langle B_n : n \in \mathbb{N} \rangle$ of finite sets such that for each $n$, $B_n \subseteq A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in B$.

$U_{fin}(A, B)$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, \ldots \in A$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in B$. 

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Many equivalences hold among these properties, and the surviving ones appear in the following Scheepers Diagram (Fig. 1) (where an arrow denotes implication) [11].

![Scheepers Diagram](image)

**Fig. 1. The Scheepers Diagram for Lindelöf space.**

The papers [11, 15, 32, 35, 36, 37, 38] have initiated the simultaneous consideration of these properties in the case where \( \mathcal{A} \) and \( \mathcal{B} \) are important families of open covers of a topological space \( X \).

In papers [1-8, 12-32] (and many others) were investigated the applications of selection principles in the study of the properties of function spaces. In particular, the properties of the space \( C_p(X) \) were investigated.

In [9] (see also [10]), Hurewicz introduced a covering property of a space \( X \), nowadays called the **Hurewicz property** in this way: for each sequence \( (U_n : n \in \mathbb{N}) \) of open covers of \( X \) there is a sequence \( (V_n : n \in \mathbb{N}) \) such that for each \( n \in \mathbb{N}, V_n \) is a finite subset of \( U_n \) and each \( x \in X \) belongs to \( \bigcup V_n \) for all but finitely many \( n \) (i.e., \( X \) satisfies \( U_{fin}(\mathcal{O}, \Gamma) \)).

In this paper we continue to study different selectors for sequences of dense sets of \( C_p(X) \) and we study a functional characterization of the covering property of Hurewicz.

**2. Main definitions and notation**

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by \( \mathbb{N} \). Let \( \mathbb{R} \) be the real line and \( \mathbb{Q} \) be the rational numbers. For a space \( X \), we denote by \( C_p(X) \) the space of all real-valued continuous functions on \( X \) with the topology of pointwise convergence.
Basic open sets of $C_p(X)$ are of the form $[x_1,...,x_k,U_1,...,U_k] = \{ f \in C(X) : f(x_i) \in U_i, i = 1,...,k \}$, where each $x_i \in X$ and each $U_i$ is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of the point $f$ as $\langle f, A, \epsilon \rangle$ where $\langle f, A, \epsilon \rangle := \{ g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in A \}$, $A$ is a finite subset of $X$ and $\epsilon > 0$.

The symbol $0$ denote the constantly zero function in $C_p(X)$. Because $C_p(X)$ is homogeneous we can work with $0$ to study local properties of $C_p(X)$.

If $X$ is a space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space $X$ is called sequentially separable if it has a countable sequentially dense set.

We recall that a subset of $X$ that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set of $X$ if $X \setminus O$ is a zero-set.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of $X$ if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space $X$ is:
- an $\omega$-cover if every finite subset of $X$ is contained in a member of $\mathcal{U}$.
- a $\gamma$-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Note that every $\gamma$-cover contains a countably $\gamma$-cover.
- a $\gamma$-shrinkable cover $\mathcal{U}$ if it is a $\gamma$-cover $\mathcal{U}$ of $X$ by co-zero sets and there exists a $\gamma$-cover $\{ F(U) : U \in \mathcal{U} \}$ of $X$ by zero-sets with $F(U) \subseteq U$ for some $U \in \mathcal{U}$.

For a topological space $X$ we denote:
- $\mathcal{O}$ — the family of all open covers of $X$;
- $\Gamma$ — the family of all countable open $\gamma$-covers of $X$;
- $\Omega$ — the family of all open $\omega$-covers of $X$;
- $\Gamma_F$ — the family of all $\gamma_F$-shrinkable covers of $X$.

For a topological space $C_p(X)$ we denote:
- $\mathcal{D}$ — the family of all dense subsets of $C_p(X)$;
- $\mathcal{S}$ — the family of all sequentially dense subsets of $C_p(X)$.

In the case of $U_{fin}$ note that for any class of covers $\mathcal{B}$ of Lindelöf space $X$, $U_{fin}(\mathcal{O}, \mathcal{B})$ is equivalent to $U_{fin}(\Gamma, \mathcal{B})$ because given an open cover $\{ U_n : n \in \mathbb{N} \}$ we may replace it by $\{ \bigcup_{i<n} U_i : n \in \mathbb{N} \}$, which is a $\gamma$-cover (unless it contains a finite subcover) of $X$.

Recall that the $i$-weight $iw(X)$ of a space $X$ is the smallest infinite cardinal number $\tau$ such that $X$ can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than $\tau$.

**Theorem 2.1.** (Noble [18]) A space $C_p(X)$ is separable iff $iw(X) = \aleph_0$. 
Let $X$ be a topological space, and $x \in X$. A subset $A$ of $X$ converges to $x$, $x = \lim A$, if $A$ is infinite, $x \notin A$, and for each neighborhood $U$ of $x$, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{ A \subseteq X : x \in \overline{A} \setminus A \}$;
- $\Gamma_x = \{ A \subseteq X : x = \lim A \}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to $x$. So, simply $\Gamma_x$ may be the set of non-trivial convergent sequences to $x$.

We write $\Pi(A_x, B_x)$ without specifying $x$, we mean $(\forall x)\Pi(A_x, B_x)$.

So we have three types of topological properties described through the selection principles:

- local properties of the form $\mathcal{S}_x(\Phi_x, \Psi_x)$;
- global properties of the form $\mathcal{S}_x(\Phi, \Psi)$;
- semi-local properties of the form $\mathcal{S}_x(\Phi, \Psi_x)$.

3. $U_{\text{fin}}(\mathcal{O}, \Omega)$

For a function space $C_p(X)$, we represent the following selection principle $F_{\text{fin}}(\mathcal{S}, D)$: whenever $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$ and a base neighborhood $\langle f, K, \epsilon \rangle$ of $f$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_k\}$ is a finite subset of $X$, there is $n' \in \mathbb{N}$ such that for each $j \in \{1, \ldots, k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

It is clear that the condition of the selection principle $F_{\text{fin}}(\mathcal{S}, D)$ can be written more briefly: whenever $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$, $\epsilon > 0$ and $K = [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that for each $x \in K$, $\min \{|f(x) - h(x)|\} < \epsilon$ for each $h \in \mathcal{F}_{n'}$.

Similarly, $F_{\text{fin}}(\Gamma_0, \Omega_0)$: whenever $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \Gamma_0$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $x \in K$, $\min \{|h(x)|\} < \epsilon$ for each $h \in \mathcal{F}_{n'}$.

**Theorem 3.1.** For a space $X$, the following statements are equivalent:

(1) $C_p(X)$ satisfies $F_{\text{fin}}(\Gamma_0, \Omega_0)$;

(2) $X$ satisfies $U_{\text{fin}}(\Gamma_F, \Omega)$.

**Proof.** (1) $\Rightarrow$ (2). Let $\{U_i\}_{i \in \mathbb{N}} \subseteq \Gamma_F$ and let $U_i = \{U_i^m\}$ for each $i \in \mathbb{N}$. We consider $\mathcal{K}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \mathbb{N}\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \mathbb{N}\}$ is a $\gamma$-cover of $X$, we have that $\mathcal{K}_i$ converges to 0 for each $i \in \mathbb{N}$. Since $C_p(X)$ satisfies $F_{\text{fin}}(\Gamma_0, \Omega_0)$, there are finite sets $\mathcal{F}_i = \{f_i^{m_1}, \ldots, f_i^{m_n(i)}\} \subseteq \mathcal{K}_i$ such that for a base neighborhood $O(f) = \langle f, K, \epsilon \rangle$ of $f = 0$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_k\}$ is a finite subset of $X$, there is $n' \in \mathbb{N}$ such that for each $j \in \{1, \ldots, k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$. Note that $\bigcup \{U_i^{m_1}, \ldots, U_i^{m_n(i)} : i \in \mathbb{N}\} \in \Omega$. 


(2) ⇒ (1). Let \( X \) satisfies \( U_{fin}(\Gamma_F, \Omega) \) and \( A_i \in \Gamma_0 \) for each \( i \in \mathbb{N} \). Consider \( \mathcal{U}_i = \{U_{i,f} = f^{-1}(\frac{1}{i+1}, \frac{1}{i}) : f \in A_i \} \) for each \( i \in \mathbb{N} \). Without loss of generality we can assume that a set \( U_{i,f} \neq X \) for any \( i \in \mathbb{N} \) and \( f \in A_i \). Otherwise there is sequence \( \{f_{k}\}_{k \in \mathbb{N}} \) such that \( \{f_{k}\}_{k \in \mathbb{N}} \) uniform converges to \( 0 \) and \( \{f_{k} : k \in \mathbb{N}\} \subseteq \Omega_0 \).

Note that \( \mathcal{F}_i = \{F_{i,m}\}_{m \in \mathbb{N}} = \{f_{i,m}^{-1}[\frac{1}{i+1}, \frac{1}{i}] : m \in \mathbb{N}\} \) is a \( \gamma \)-cover of \( X \) and \( F_{i,m} \subseteq U_{i,m} \) for each \( i, m \in \mathbb{N} \). It follows that \( \mathcal{U}_i \subseteq \Gamma_F \) for each \( i \in \mathbb{N} \).

Since \( X \) satisfies \( U_{fin}(\Gamma_F, \Omega) \), there is a sequence \( \{U_{i,m(1)}, U_{i,m(2)}, \ldots, U_{i,m(i)} : i \in \mathbb{N}\} \) such that for each \( i \) and \( k \in \{m(1), \ldots, m(i)\} \), \( U_{i,m(k)} \in \mathcal{U}_i \) and
\[
\left\{\bigcup_{k=m(1)}^{m(i)} U_{i,k} : i \in \mathbb{N} \right\} \subseteq \Omega.
\]

Let \( (0, K, \epsilon) \) be a base neighborhood of \( 0 \) where \( \epsilon > 0 \) and \( K = \{x_1, \ldots, x_s\} \) is a finite subset of \( X \), then there is \( i_1 \in \mathbb{N} \) such that \( \frac{1}{i_1} < \epsilon \) and \( \bigcup_{k=m(1)}^{m(i_1)} U_{i_1,k} \supseteq K \). It follows that for each \( j \in \{1, \ldots, s\} \) there is \( g \in \{f_{i_1,m(1)}, \ldots, f_{i_1,m(i_1)}\} \) such that \( g(x_j) \in (-\epsilon, \epsilon) \).

\[ \square \]

**Lemma 3.2.** (Lemma 6.5 in [20]) Let \( \mathcal{U} = \{U_n : n \in \mathbb{N}\} \) be a \( \gamma_F \)-shrinkable co-zero cover of a space \( X \). Then the set \( S = \{f \in C(X) : f \upharpoonright (X \setminus U_n) \equiv 1 \) for some \( n \in \mathbb{N}\} \) is sequentially dense in \( C_p(X) \).

**Theorem 3.3.** For a space \( X \) with \( iw(X) = \aleph_0 \), the following statements are equivalent:

1. \( C_p(X) \) satisfies \( F_{fin}(\mathcal{S}, \mathcal{D}) \);
2. \( X \) satisfies \( U_{fin}(\Gamma_F, \Omega) \);
3. \( C_p(X) \) satisfies \( F_{fin}(\Gamma_0, \Omega_0) \);
4. \( C_p(X) \) satisfies \( F_{fin}(\mathcal{S}, \Omega_0) \).

**Proof.** (1) ⇒ (2). Let \( \mathcal{U}_i = \{U^j_i : j \in \mathbb{N}\} \subseteq \Gamma_F \) for each \( i \in \mathbb{N} \). Then, by Lemma 3.2, each \( S_i = \{f \in C(X) : f \upharpoonright (X \setminus U^j_i) \equiv 1 \) for some \( m \in \mathbb{N}\} \) is sequentially dense in \( C_p(X) \). Since \( C_p(X) \) satisfies \( F_{fin}(\mathcal{S}, \mathcal{D}) \), there are finite sets \( F_i = \{f_1^{m_1}, \ldots, f_i^{m(i)}\} \subseteq S_i \) such that for each \( f \in C_p(X) \) and a base neighborhood \( \langle f, K, \epsilon \rangle \) of \( f \) where \( \epsilon > 0 \) and \( K = \{x_1, \ldots, x_k\} \) is a finite subset of \( X \), there is \( n' \in \mathbb{N} \) such that for each \( j \in \{1, \ldots, k\} \) there is \( g \in F_{n'} \) such that \( g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon) \).

(2) ⇒ (3). By Theorem 3.1.

(3) ⇒ (4) is immediate.

(4) ⇒ (1). Suppose that \( C_p(X) \) satisfies \( F_{fin}(\mathcal{S}, \Omega_0) \).

Let \( D = \{d_n : n \in \mathbb{N}\} \) be a dense subspace of \( C_p(X) \) and \( S_i \subseteq \mathcal{S} \) for each \( i \in \mathbb{N} \). Given a sequence of sequentially dense subspace of \( C_p(X) \), enumerate it as \( \{S_{n,m} : n, m \in \mathbb{N}\} \). For each \( n \in \mathbb{N} \), pick \( d_{n,m} \in S_{n,m} \) so that for a base neighborhood \( \langle d_n, K, \epsilon \rangle \) of \( d_n \) where \( \epsilon > 0 \) and \( K = \{x_1, \ldots, x_k\} \) is a finite subset of \( X \), there is \( m' \in \mathbb{N} \) such that for each \( j \in \{1, \ldots, k\} \) there is \( f \in F_{n,m'} \) such that \( g(x_j) \in (d_n(x_j) - \epsilon, d_n(x_j) + \epsilon) \). It follows that \( C_p(X) \) satisfies \( F_{fin}(\mathcal{S}, \mathcal{D}) \). \[ \square \]
Theorem 3.4. For a space $X$ the following statements are equivalent:

(1) $X$ is Lindel"of and $X$ satisfies $U_{fin}(\Gamma_F, \Omega)$;
(2) $X$ satisfies $U_{fin}(\mathcal{O}, \Omega)$.

Proof. It is proved similarly to the proof of Theorem 4.1. □

Theorem 3.5. For a separable metrizable space $X$, the following statements are equivalent:

(1) $C_p(X)$ satisfies $F_{fin}(\mathcal{S}, D)$;
(2) $X$ satisfies $U_{fin}(\mathcal{O}, \Omega)$;
(3) $C_p(X)$ satisfies $F_{fin}(\Gamma_0, \Omega_0)$;
(4) $C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \Omega_0)$.

4. $U_{fin}(\mathcal{O}, \Gamma)$ - Hurewicz property

Theorem 4.1. For a space $X$ the following statements are equivalent:

(1) $X$ satisfies $U_{fin}(\Gamma_F, \Gamma)$ and is Lindel"of;
(2) $X$ has the Hurewicz property.

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X$. For every $n$, $U \in \mathcal{U}_n$ and $x \in X$ we find co-zero sets $W_{0,n,U,x} \subset W_{2,n,U,x}$, and a zero-set $W_{1,n,U,x}$ such that $x \in W_{0,n,U,x} \subset W_{1,n,U,x} \subset W_{2,n,U,x} \subset U$. Since $X$ is Lindel"of, there is a sequence $(x^n_k : k \in \mathbb{N})$ such that $X$ is covered by $\{W_{0,n,U,x^n_k} : k \in \mathbb{N}\}$. Look at the cover $\mathcal{W}_n$ of $X$ consisting of sets $W^n_k = \bigcup_{i \leq k} W_{2,n,U,x^n_i}$, $k \in \mathbb{N}$. Note that $\mathcal{W}_n \in \Gamma_F$ because $\bigcup_{i \leq k} W_{1,n,U,x^n_i}$ is a zero-set contained in $W^n_k$, and $\{\bigcup_{i \leq k} W_{1,n,U,x^n_i} : k \in \mathbb{N}\}$ is a $\gamma$-cover of $X$ because even $\{\bigcup_{i \leq k} W_{0,n,U,x^n_i} : k \in \mathbb{N}\}$ is a $\gamma$-cover of $X$.

Now use the property $U_{fin}(\Gamma_F, \Gamma)$ to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ together with the fact that $\mathcal{W}_n$ is a finer cover that $\mathcal{U}_n$ for all $n$. □

For a function space $C_p(X)$, we represent the following selection principle $F_{fin}(\mathcal{S}, \mathcal{S})$: whenever $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$ there is $\{\mathcal{F}_{nk} : k \in \mathbb{N}\}$ such that for a base neighborhood $(f, K, \epsilon)$ of $f$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_m\}$ is a finite subset of $X$, there is $k' \in \mathbb{N}$ such that for each $k > k'$ and $\forall j \in \{1, \ldots, m\}$ there is $g \in \mathcal{F}_{nk}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

It is clear that the condition of the selection principle $F_{fin}(\mathcal{S}, \mathcal{S})$ can be written more briefly: whenever $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$, $\epsilon > 0$ and $K \in [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that for every $n > n'$ $\min_{h \in \mathcal{F}_n} \{||f(x) - h(x)||\} < \epsilon$ for each $x \in K$.

Similarly, $F_{fin}(\Gamma_0, \Gamma_0)$: whenever $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots \in \Gamma_0$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for $\epsilon > 0$ and $K \in [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that for every $n > n'$ $\min_{h \in \mathcal{F}_n} \{|h(x)|\} < \epsilon$ for each $x \in K$.

Theorem 4.2. For a space $X$, the following statements are equivalent:
(1) \( C_p(X) \) satisfies \( F_{\text{fin}}(\Gamma_0, \Gamma_0) \);
(2) \( X \) satisfies \( U_{\text{fin}}(\Gamma_F, \Gamma) \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( \{ U_i \}_{i \in \mathbb{N}} \subset \Gamma_F, \ U_i = \{ U_i^m \}_{m \in \mathbb{N}} \) for each \( i \in \mathbb{N} \). We consider a subset \( S_i \) of \( C_p(X) \) where
\[
S_i = \{ f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0 \text{ and } f_i^m \upharpoonright (X \setminus U_i^m) = 1 \text{ for } m \in \mathbb{N} \}.
\]
Since \( F_i = \{ F(U_i^m) : m \in \mathbb{N} \} \) is a \( \gamma \)-cover of \( X \), we have that \( S_i \) converges to \( 0 \), i.e. \( S_i \in \Gamma_0 \) for each \( i \in \mathbb{N} \).

Since \( C(X) \) satisfies \( F_{\text{fin}}(\Gamma_0, \Gamma_0) \), there is a sequence \( \{ F_i \}_{i \in \mathbb{N}} = \{ f_i^{m_1}, \ldots, f_i^{m_{k(i)}} : i \in \mathbb{N} \} \) such that for each \( i \), \( F_i \subseteq S_i \), and for a base neighborhood \( \langle 0, K, \epsilon \rangle \) of \( 0 \) where \( \epsilon > 0 \) and \( K = \{ x_1, \ldots, x_k \} \) is a finite subset of \( X \), there is \( n' \in \mathbb{N} \) such that for each \( n > n' \) and \( j \in \{ 1, \ldots, k \} \) there is \( g \in F_n \) such that \( g(x_j) \in (-\epsilon, \epsilon) \).

Consider the sequence \( \{ W_i \}_{i \in \mathbb{N}} = \{ U_i^{m_1}, \ldots, U_i^{m_{k(i)}} : i \in \mathbb{N} \} \).

(a) \( W_i \subset U_i \) for each \( i \in \mathbb{N} \).

(b) \( \bigcup \{ W_i : i \in \mathbb{N} \} \) is a \( \gamma \)-cover of \( X \).

Let \( K = \{ x_1, \ldots, x_s \} \) be a finite subset of \( X \) and \( \langle 0, K, \frac{1}{2} \rangle \) be a base neighborhood of \( 0 \), then there exists \( i_0 \in \mathbb{N} \) such that for each \( i > i_0 \) and \( j \in \{ 1, \ldots, s \} \) there is \( x_j \in F_i \) such that \( g(x_j) \in (-\frac{1}{2}, \frac{1}{2}) \).

It follows that \( K \subset \bigcup_{j=1}^{k(i)} U_i^{m_j} \) for \( i > i_0 \). We thus get that \( X \) satisfies \( U_{\text{fin}}(\Gamma_F, \Gamma) \).

(2) \( \Rightarrow \) (1). Fix \( \{ S_i : i \in \mathbb{N} \} \subset \Gamma_0 \) where \( S_i = \{ f_i^k : k \in \mathbb{N} \} \) for each \( i \in \mathbb{N} \).

For each \( i, k \in \mathbb{N} \), we put \( U_{i,k} = \{ x \in X : |f_i^k(x)| < \frac{1}{k} \} \), \( Z_{i,k} = \{ x \in X : |f_i(x)| \leq \frac{1}{k} \} \).

Each \( U_{i,k} \) (resp., \( Z_{i,k} \)) is a cozero-set (resp., zero-set) in \( X \) with \( Z_{i,k} \subset U_{i,k} \). Let \( U_i = \{ U_{i,k} : k \in \mathbb{N} \} \) and let \( Z_{i,k} = \{ Z_{i,k} : k \in \mathbb{N} \} \). So without loss of generality, we may assume \( U_{i,k} \neq X \) for each \( i, k \in \mathbb{N} \). We can easily check that the condition \( f_i^k \rightarrow 0 \) \( (k \rightarrow \infty) \) implies that \( Z_i \) is a \( \gamma \)-cover of \( X \).

Since \( X \) satisfies \( U_{\text{fin}}(\Gamma_F, \Gamma) \) there is a sequence \( \{ F_i \}_{i \in \mathbb{N}} = \{ U_{i,k_1}, \ldots, U_{i,k_i} : i \in \mathbb{N} \} \) such that for each \( i \), \( F_i \subseteq U_i \), and \( \{ \bigcup F_i : i \in \mathbb{N} \} \) is an element of \( \Gamma \).

Let \( K = \{ x_1, \ldots, x_s \} \) be a finite subset of \( X \), \( \epsilon > 0 \), and \( \langle 0, K, \epsilon \rangle \) be a base neighborhood of \( 0 \), then there exists \( i' \in \mathbb{N} \) such that for every \( i > i' \) \( K \subset \bigcup F_i \). It follow that for every \( i > i' \) and \( j \in \{ 1, \ldots, s \} \) there is \( g \in S_i \) such that \( g(x_j) \in (-\epsilon, \epsilon) \). So \( C_p(X) \) satisfies \( F_{\text{fin}}(\Gamma_0, \Gamma_0) \). \( \square \)

**Theorem 4.3.** For a Lindelöf space \( X \), the following statements are equivalent:

(1) \( C_p(X) \) satisfies \( F_{\text{fin}}(\Gamma_0, \Gamma_0) \);
(2) \( X \) has the Hurewicz property.

A space \( X \) has *Velichko property* \( (X \models V) \), if there exists a condensation (one-to-one continuous mapping) \( f : X \rightarrow Y \) from the space \( X \) on a separable metric space \( Y \), such that \( f(U) \) is an \( F_\sigma \)-set of \( Y \) for any cozero-set \( U \) of \( X \).
Theorem 4.4. (Velichko [40]). Let $X$ be a space. A space $C_p(X)$ is sequentially separable iff $X \models V$.

Theorem 4.5. For a space $X$ with $X \models V$, the following statements are equivalent:

1. $C_p(X)$ satisfies $F_{fin}(S,S)$;
2. $X$ satisfies $U_{fin}(\Gamma_F, \Gamma)$;
3. $C_p(X)$ satisfies $F_{fin}(\Gamma_0, \Gamma_0)$;
4. $C_p(X)$ satisfies $F_{fin}(S, \Gamma_0)$.

Proof. (1) $\Rightarrow$ (2). Let $U_i = \{U_i^j : j \in \mathbb{N}\} \in \Gamma_F$ for each $i \in \mathbb{N}$. Then, by Lemma 3.2, each $S_i = \{f \in C(X) : f \mid (X \setminus U_i^j) \equiv 1 \text{ for some } m \in \mathbb{N}\}$ is sequentially dense in $C_p(X)$.

Since $C(X)$ satisfies $U_{fin}(S,S)$, there is a sequence $\{F_i\} = \{f_i^{m_1}, \ldots, f_i^{m_s} : i \in \mathbb{N}\}$ such that for $f = 0$ there is $F_{ik} : k \in \mathbb{N}$ such that for a base neighborhood $(f, K, \epsilon)$ of $f$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_m\}$ is a finite subset of $X$, there is $k' \in \mathbb{N}$ such that for each $k > k'$ and $j \in \{1, \ldots, m\}$ there is $g \in F_{ik}$ such that $g(x_j) \in (-\epsilon, \epsilon)$.

Let $\epsilon = \frac{1}{2}$ and $\mathbb{N}' = \mathbb{N}\setminus \{k'\}$. Consider a sequence $\{Q_k\}_{k \in \mathbb{N}'} = \{U_{ik}^{m_1}, \ldots, U_{ik}^{m_s} : k \in \mathbb{N}'\}$ for corresponding to $\{F_{ik}\} = \{f_{ik}^{m_1}, \ldots, f_{ik}^{m_s} : k \in \mathbb{N}'\}$.

(a). $Q_k \subset U_{ik}$ for $k \in \mathbb{N}'$.

(b). $\bigcup Q_k : k \in \mathbb{N}'$ is a $\gamma$-cover of $X$. We thus get $X$ satisfies $U_{fin}(\Gamma_F, \Gamma)$.

(2) $\Rightarrow$ (3). By Theorem 4.2.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). For each $n \in \mathbb{N}$, let $S_n$ be a sequentially dense subset of $C_p(X)$ and let $\{h_n : n \in \mathbb{N}\}$ be sequentially dense in $C_p(X)$. Take a sequence $\{f_n^m : m \in \mathbb{N}\} \subset S_n$ such that $f_n^m \to h_n (m \to \infty)$. Then $f_n^m - h_n \to 0 (m \to \infty)$. Hence, there exist $\mathcal{F}_n = \{f_n^{m_1}, \ldots, f_n^{m_s}\} \subset S_n$ such that $\bigcup \{f_n^{m_1} - h_n, \ldots, f_n^{m_s} - h_n\} : n \in \mathbb{N}\} \in \Gamma_0$, i.e. for a base neighborhood $(f, K, \epsilon)$ of $f = 0$ where $\epsilon > 0$ and $K = \{x_1, \ldots, x_m\}$ is a finite subset of $X$, there is $n' \in \mathbb{N}$ such that for each $n > n'$ and $\forall j \in \{1, \ldots, m\}$ there is $g \in \{f_n^{m_1} - h_n, \ldots, f_n^{m_s} - h_n\}$ such that $g(x_j) \in (-\epsilon, \epsilon)$.

Let $h \in C_p(X)$ and take a sequence $\{h_{n_j} : j \in \mathbb{N}\} \subset \{h_n : n \in \mathbb{N}\}$ converging to $h$. Let $K = \{x_1, \ldots, x_m\}$ be a finite subset of $X$ and $\epsilon > 0$. Consider a base neighborhood $(h, K, \epsilon)$ of $h$. Then there is $j' \in \mathbb{N}$ such that $h_{n_j} \in (h, K, \epsilon)$ and $\forall s \in \{1, \ldots, m\}$ there is $g \in \{f_{n_j}^{m_1} - h_{n_j}, \ldots, f_{n_j}^{m_s} - h_{n_j}\}$ such that $g(x_s) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ for $j > j'$. It follows that for each $s \in \{1, \ldots, m\}$ there is $l(j) \in \Gamma h(n_j)$ such that $(f_{n_j}^{(m_1)} - h_{n_j}) + (h_{n_j} - h)(x_s) \in (-\epsilon, \epsilon)$ for $j > j'$. Hence $C_p(X)$ satisfies $F_{fin}(S, S)$.

Theorem 4.6. For a separable metrizable space $X$, the following statements are equivalent:


(1) $C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \mathcal{S})$;
(2) $X$ satisfies $U_{fin}(\mathcal{O}, \Gamma)$ [Hurewicz property];
(3) $C_p(X)$ satisfies $F_{fin}(\Gamma_0, \Gamma_0)$;
(4) $C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \Gamma_0)$.

Recall that a space $X$ is said to be Rothberger [27] (or, [17]) if for every sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$, there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n$, $V_n \in U_n$, and $\{V_n : n \in \mathbb{N}\}$ is an open cover of $X$.

A space $X$ is said to be Menger [9] (or, [30]) if for every sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$, there are finite subfamilies $V_n \subset U_n$ such that $\bigcup \{V_n : n \in \mathbb{N}\}$ is a cover of $X$.

Every $\sigma$-compact space is Menger, and a Menger space is Lindelöf.

In [21], we gave the functional characterizations of Rothberger and Menger properties.

Recall that if $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic, uniformly homeomorphic), we say that the spaces $X$ and $Y$ are $t$-equivalent ($l$-equivalent, $u$-equivalent). The properties preserved by $t$-equivalence ($l$-equivalence, $u$-equivalence) we call $t$-invariant ($l$-invariant, $u$-invariant) [2].

**Question 1.** Is the Hurewicz (Rothberger, Menger) property $t$-invariant? $l$-invariant? $u$-invariant?

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**REFERENCES**