

Coincidence Quasi-Best Proximity Points for Quasi-Cyclic-Noncyclic Mappings in Convex Metric Spaces

Ali Abkar*, Masoud Norouzian

Department of Pure Mathematics, Faculty of Science,
Imam Khomeini International University, Qazvin 34149, Iran

E-mail: abkar@sci.ikiu.ac.ir

E-mail: norouzian.m67@gmail.com

ABSTRACT. We introduce the notion of quasi-cyclic-noncyclic pair and its relevant new notion of coincidence quasi-best proximity points in a convex metric space. In this way we generalize the notion of coincidence-best proximity point already introduced by M. Gabeleh et al [14]. It turns out that under some circumstances this new class of mappings contains the class of cyclic-noncyclic mappings as a subclass. The existence and convergence of coincidence-best and coincidence quasi-best proximity points in the setting of convex metric spaces are investigated.

Keywords: Coincidence-best proximity point, Cyclic-noncyclic contraction, Quasi-cyclic-noncyclic contraction, Uniformly convex metric space.

2010 Mathematics subject classification: 47H10, 47H09, 46B20.

1. INTRODUCTION

Let (X, d) be a metric space, and let A, B be subsets of X . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$; similarly, a mapping $S : A \cup B \rightarrow A \cup B$ is said to be *noncyclic* if $S(A) \subseteq A$ and $S(B) \subseteq B$. The following theorem is an extension of Banach contraction principle.

*Corresponding Author

Theorem 1.1. ([18]) *Let A and B be nonempty closed subsets of a complete metric space (X, d) . Suppose that T is a cyclic mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y),$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$. Then T has a unique fixed point in $A \cap B$.

Let A and B be nonempty subsets of a metric space X . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a *cyclic contraction* if T is cyclic and

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) \text{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$, where

$$\text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

For a cyclic mapping $T : A \cup B \rightarrow A \cup B$, a point $x \in A \cup B$ is said to be a best proximity point provided that

$$d(x, Tx) = \text{dist}(A, B).$$

The following existence, uniqueness and convergence result of a best proximity point for cyclic contractions is the main result of [8].

Theorem 1.2. ([8]) *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $T : A \cup B \rightarrow A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \rightarrow x$ and*

$$\|x - Tx\| = \text{dist}(A, B).$$

In the theory of best proximity points, one usually considers a cyclic mapping T defined on the union of two (closed) subsets of a given metric space. Here the objective is to minimize the expression $d(x, Tx)$ where x runs through the domain of T ; that is $A \cup B$. In other words, we want to find

$$\min\{d(x, Tx) : x \in A \cup B\}.$$

If A and B intersect, the solution is clearly a fixed point of T ; otherwise we have

$$d(x, Tx) \geq \text{dist}(A, B), \quad \forall x \in A \cup B,$$

so that the point at which the equality occurs is called a best proximity point of T . This point of view dominates the literature.

Very recently, M. Gabeleh, O. Olela Otafudu, and N. Shahzad [14] considered two mappings T and S simultaneously and established interesting results. For technical reasons, the first map should be cyclic and the second one should be noncyclic. According to [14], for a nonempty pair of subsets (A, B) , and a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ (that is, $T : A \cup B \rightarrow A \cup B$ is cyclic and

$S : A \cup B \rightarrow A \cup B$ is noncyclic); they called a point $p \in A \cup B$ a *coincidence best proximity point* for $(T; S)$ provided that

$$d(Sp, Tp) = \text{dist}(A, B).$$

Note that if $S = I$, the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for T . Also, if $\text{dist}(A, B) = 0$, then p is called a *coincidence point* for $(T; S)$ (see [12] and [15] for more information). With the definition just given, and depending on the situation as to whether S equals the identity map, or if the distance between the underlying sets is zero, one obtains a best proximity point for T , or a coincidence point for the pair $(T; S)$. This was in fact the philosophy behind the phrase *coincidence-best proximity point* coined by Gabeleh et al. They then defined the notion of a cyclic-noncyclic contraction.

Definition 1.3. ([14]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : A \cup B \rightarrow A \cup B$ be two mappings. The pair $(T; S)$ is called a cyclic-noncyclic contraction pair if it satisfies the following conditions:

- (1) $(T; S)$ is a cyclic-noncyclic pair on $A \cup B$.
- (2) For some $r \in (0, 1)$ we have

$$d(Tx, Ty) \leq rd(Sx, Sy) + (1 - r)\text{dist}(A, B), \quad \forall(x, y) \in A \times B.$$

To state the main result of [14], we need to recall the notion of convexity in the framework of metric spaces. In [26], Takahashi introduced the notion of convexity in metric spaces as follows (see also [24]).

Definition 1.4. Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \rightarrow X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure \mathcal{W} is called a *convex metric space*, and is denoted by (X, d, \mathcal{W}) . A Banach space and each of its convex subsets are convex metric spaces.

A subset K of a convex metric space (X, d, \mathcal{W}) is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Similarly, a convex metric space (X, d, \mathcal{W}) is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$,

$$d(z, \mathcal{W}(x, y; \frac{1}{2})) \leq r(1 - \alpha) < r.$$

For example every uniformly convex Banach space is a uniformly convex metric space.

Definition 1.5. ([14]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d) . A mapping $S : A \cup B \rightarrow A \cup B$ is said to be a relatively anti-Lipschitzian mapping if there exists $c > 0$ such that

$$d(x, y) \leq cd(Sx, Sy), \quad \forall (x, y) \in A \times B.$$

The main result of M. Gabeleh et al reads as follows:

Theorem 1.6. ([14]) Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space (X, d, \mathcal{W}) such that A is convex. Let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$ such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then $(T; S)$ has a coincidence best proximity point in A . Further, if $x_0 \in A$ and $Sx_{n+1} := Tx_n$, then (x_{2n}) converges to the coincidence-best proximity point of $(T; S)$.

Existence of best proximity pairs was first studied in [9] by using a geometric property on a nonempty pair of subsets of a Banach space, called *proximal normal structure*, for noncyclic relatively nonexpansive mappings (Theorem 2.2 of [9]). Some existence results of best proximity pairs can be found in [1, 2, 5, 6, 7, 10, 11, 13, 17, 23, 25].

In the current paper, we study sufficient conditions which ensure the existence and convergence of *coincidence-best and quasi-best proximity point* for a pair of quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

2. COINCIDENCE QUASI-BEST PROXIMITY POINT

In this section, we introduce the class of quasi-cyclic-noncyclic mappings that contains the class of cyclic-noncyclic mappings as a subclass. Next, we introduce the new notion of quasi-best proximity points for this mappings. Finally, we study the existence and convergence of coincidence quasi-best proximity points for quasi-cyclic-noncyclic contraction mappings in the setting of convex metric spaces.

Definition 2.1. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \rightarrow X$ be two mappings. The pair $(T; S)$ is called a quasi-cyclic-noncyclic (**QCN**) contraction pair if it satisfies the following conditions:

- (1) $(T; S)$ is a quasi-cyclic-noncyclic pair on X ; that is,

$$T(A) \subseteq S(B), \quad T(B) \subseteq S(A).$$

- (2) For some $\alpha \in (0, 1)$ and for each $(x, y) \in A \times B$ we have

$$d(Tx, Ty) \leq \alpha d(Sx, Sy) + (1 - \alpha) \text{dist}(S(A), S(B)).$$

Note that if $S(A) = A$ and $S(B) = B$, then the above definition reduces to Definition 1.3; that is, the pair $(T; S)$ is a cyclic-noncyclic pair.

EXAMPLE 2.2. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} -x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x + 1, & \text{if } x \in A \\ 2x - 1, & \text{if } x \in B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{2}(2y - 2x - 2) + \frac{1}{2}(2) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)). \end{aligned}$$

Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$.

The next example shows that there is a QCN mapping that is not a cyclic-noncyclic mapping.

EXAMPLE 2.3. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} -x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} x + 1, & \text{if } x \in A \\ x - 1, & \text{if } x \in B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a quasi-cyclic-noncyclic pair that is not a cyclic-noncyclic pair.

Remark 2.4. Notice that (2) implies that

$$d(Tx, Ty) \leq d(Sx, Sy), \quad \forall (x, y) \in A \times B.$$

Moreover, if S is a noncyclic relatively nonexpansive mapping; meaning that

$$d(Sx, Sy) \leq d(x, y), \quad \forall (x, y) \in A \times B,$$

then T is a cyclic contraction. In addition, if in the above definition S is assumed to be continuous, then T would be continuous too.

Definition 2.5. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \rightarrow X$ be a quasi-cyclic-noncyclic pair on X . A point $p \in A \cup B$ is said to be a coincidence quasi-best proximity point for $(T; S)$ provided that

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Note that if $S = I$, then p reduces to a coincidence-best proximity point for $(T; S)$.

To prove the main result of this section, we need some preparations.

Lemma 2.6. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a quasi-cyclic-noncyclic pair defined on X . Then there exists a sequence $\{x_n\}$ in X such that for all $n \geq 0$ we have $Tx_n = Sx_{n+1}$ where $\{x_{2n}\}, \{x_{2n+1}\}$ are subsequences in A and B respectively.*

Proof. Let $x_0 \in A$. Since $Tx_0 \in S(B)$, there exists $x_1 \in B$ such that $Tx_0 = Sx_1$. Again, since $Tx_1 \in S(A)$, there exists $x_2 \in A$ such that $Tx_1 = Sx_2$.

Continuing this process, we obtain a sequence $\{x_n\}$, such that $\{x_{2n}\}, \{x_{2n+1}\}$ are in A and B respectively and $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. \square

Lemma 2.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X . For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then we have*

$$d(Sx_{2n}, Sx_{2n+1}) \rightarrow \text{dist}(S(A), S(B)).$$

Proof.

$$\begin{aligned} d(Sx_{2n+1}, Sx_{2n+2}) &= d(Tx_{2n}, Tx_{2n+1}) \\ &\leq \alpha d(Sx_{2n}, Sx_{2n+1}) + (1 - \alpha) \text{dist}(S(A), S(B)) \\ &= \alpha d(Tx_{2n-1}, Tx_{2n}) + (1 - \alpha) \text{dist}(S(A), S(B)) \\ &\leq \alpha [\alpha d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha) \text{dist}(S(A), S(B))] \\ &\quad + (1 - \alpha) \text{dist}(S(A), S(B)) \\ &= \alpha^2 d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha^2) \text{dist}(S(A), S(B)) \\ &= \alpha^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1 - \alpha^2) \text{dist}(S(A), S(B)) \\ &\leq \dots \\ &\leq \alpha^{2n} d(Tx_0, Tx_1) + (1 - \alpha^2) \text{dist}(S(A), S(B)). \end{aligned}$$

Now, if $n \rightarrow \infty$ in above relation, we conclude that

$$d(Sx_{2n}, Sx_{2n+1}) \rightarrow \text{dist}(S(A), S(B)).$$

\square

Theorem 2.8. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X . Assume that S is continuous on A . For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A , then the pair $(T; S)$ has a coincidence quasi-best proximity point in A .*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \rightarrow p \in A$. We have

$$\begin{aligned} \text{dist}(S(A), S(B)) &\leq d(Tx_{2n_k-1}, Tp) \leq d(Sx_{2n_k-1}, Sp) \\ &\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}). \end{aligned}$$

By Lemma 2.7, if $k \rightarrow \infty$, we obtain that

$$d(Tx_{2n_k-1}, Tp) \rightarrow \text{dist}(S(A), S(B)).$$

Moreover, we have

$$\begin{aligned} \text{dist}(S(A), S(B)) &\leq d(Sp, Tp) \\ &\leq d(Sp, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tp) \\ &= d(Sp, Sx_{2n_k}) + d(Tx_{2n_k-1}, Tp) \\ &\rightarrow \text{dist}(S(A), S(B)), \end{aligned}$$

that is,

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

□

Lemma 2.9. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN contraction pair defined on X . For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then $\{Sx_{2n}\}$, and $\{Sx_{2n+1}\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.*

Proof. Since

$$d(Sx_{2n}, Sx_{2n+1}) \rightarrow \text{dist}(S(A), S(B)),$$

it suffices to show that $\{Sx_{2n}\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_2, Sx_{2N_0+1}) > M, \quad d(Sx_2, Sx_{2N_0-1}) \leq M,$$

where,

$$M > \max \left\{ \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \text{dist}(S(A), S(B)), \quad d(Sx_1, Sx_0) \right\}.$$

By the above assumption, we have

$$\begin{aligned} &\frac{M - \text{dist}(S(A), S(B))}{\alpha^2} + \text{dist}(S(A), S(B)) \\ &< \frac{d(Sx_2, Sx_{2N_0+1}) - \text{dist}(S(A), S(B))}{\alpha^2} \\ &+ \text{dist}(S(A), S(B)) \\ &\leq \frac{d(Sx_2, Sx_{2N_0+1}) + (\alpha^2 - 1)d(Sx_2, Sx_{2N_0+1})}{\alpha^2} \\ &= d(Sx_2, Sx_{2N_0+1}) = d(Tx_1, Tx_{2N_0}) \\ &\leq d(Sx_1, Sx_{2N_0}) = d(Tx_0, Tx_{2N_0-1}) \\ &= d(Sx_0, Sx_{2N_0-1}) \\ &\leq d(Sx_0, Sx_2) + d(Sx_2, Sx_{2N_0-1}) \\ &\leq d(Sx_0, Sx_2) + M. \end{aligned}$$

This implies that

$$\frac{M - \text{dist}(S(A), S(B))}{\alpha^2} + \text{dist}(S(A), S(B)) < d(Sx_0, Sx_2) + M,$$

hence,

$$M - (1 - \alpha^2)\text{dist}(S(A), S(B)) < \alpha^2[d(Sx_0, Sx_2) + M],$$

and,

$$(1 - \alpha^2)M < \alpha^2 d(Sx_0, Sx_2) + (1 - \alpha^2)\text{dist}(S(A), S(B)).$$

Now, it follows that

$$M < \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \text{dist}(S(A), S(B)),$$

which contradicts the choice of M . \square

Before we state the following theorem, we recall that a subset $A \subseteq X$ is said to be boundedly compact if the closure of every bounded subset of A is compact and is contained in A .

Theorem 2.10. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that $S(A)$ is boundedly compact and let $(T; S)$ be a QCN contraction pair defined on X . If S is relatively anti-Lipschitzian and continuous on A , then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Proof. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. By Lemma 2.9, $\{Sx_{2n}\}$ is bounded in $S(A)$. On the other hand, $S(A)$ is boundedly compact, so that there exists a subsequence $\{Sx_{2n_k}\}$ of $\{Sx_{2n}\}$ such that

$$Sx_{2n_k} \rightarrow Sp,$$

for some $p \in A$. We know that S is relatively anti-Lipschitzian, therefore

$$d(x_{2n_k}, p) \leq c d(Sx_{2n_k}, Sp) \rightarrow 0, \quad k \rightarrow \infty.$$

This implies that $\{x_{2n_k}\}$ is a convergent subsequence of $\{x_{2n}\}$. Now, the result follows from Theorem 2.8. \square

EXAMPLE 2.11. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, 0]$ and $B = [0, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} -x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{2}(2y - 2x) + \frac{1}{2}(0) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)). \end{aligned}$$

Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$. Moreover, S is continuous on A and $S(A)$ is boundedly compact in X . Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x \geq |x - y|.$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(S(A), S(B)) = 0 \text{ or } -p - 2p = 0,$$

which implies that $p = 0$. In this case, p is a fixed point of S .

In the following we supply an example which shows that there exists a coincidence quasi-best proximity point that is not a fixed point of S .

EXAMPLE 2.12. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, 0]$ and $B = [0, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} -(x+1), & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a QCN contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{2}(2y - 2x) + \frac{1}{2}(0) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(S(A), S(B)). \end{aligned}$$

Also, $T(A) = [1, +\infty) \subseteq S(B)$ and $T(B) = (-\infty, -1] \subseteq S(A)$. Moreover, S is continuous on A and $S(A)$ is boundedly compact in X . Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x \geq |x - y|.$$

Finally, the existence of coincidence quasi-best proximity point of the pair $(T; S)$ follows from Theorem 2.10; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(S(A), S(B)) = 0 \text{ or } -(p+1) - 2p = 0,$$

which implies that $p = -\frac{1}{3}$.

Lemma 2.13. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space (X, d, \mathcal{W}) such that $S(A)$ is convex. Let $(T; S)$ be a QCN contraction pair defined on X . For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \rightarrow 0, \quad d(Sx_{2n+3}, Sx_{2n+1}) \rightarrow 0.$$

Proof. We prove that $d(Sx_{2n+2}, Sx_{2n}) \rightarrow 0$. To the contrary, assume that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ such that

$$d(Sx_{2n_k+2}, Sx_{2n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(S(A), S(B))$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - \text{dist}(S(A), S(B)), \frac{\text{dist}(S(A), S(B))\alpha(\gamma)}{1 - \alpha(\gamma)} \right\}.$$

By Lemma 2.7, since $d(Sx_{2n_k}, Sx_{2n_k+1}) \rightarrow \text{dist}(S(A), S(B))$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} d(Sx_{2n_k}, Sx_{2n_k+1}) &\leq \text{dist}(S(A), S(B)) + \varepsilon, \\ d(Sx_{2n_k+2}, Sx_{2n_k+1}) &\leq \text{dist}(S(A), S(B)) + \varepsilon \end{aligned}$$

and

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \geq \varepsilon_0 > \gamma(\text{dist}(S(A), S(B)) + \varepsilon).$$

It now follows from the uniform convexity of X and the convexity of $S(A)$ that

$$\begin{aligned} \text{dist}(S(A), S(B)) &\leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \frac{1}{2})) \\ &\leq (\text{dist}(S(A), S(B)) + \varepsilon)(1 - \alpha(\gamma)) \\ &< \text{dist}(S(A), S(B)) + \frac{\text{dist}(S(A), S(B))\alpha(\gamma)}{1 - \alpha(\gamma)}(1 - \alpha(\gamma)) \\ &= \text{dist}(S(A), S(B)), \end{aligned}$$

which is a contradiction. Similarly, we see that $d(Sx_{2n+3}, Sx_{2n+1}) \rightarrow 0$. \square

The following Theorem guarantees the existence and convergence of coincidence quasi-best proximity points for QCN contraction mappings in the setting of uniformly convex metric spaces.

Theorem 2.14. *Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d; \mathcal{W})$ such that $S(A)$ is convex. Let $(T; S)$ be a QCN contraction pair defined on X such that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(S(A), S(B)).$$

Further, if $x_0 \in A$ and $Tx_n = Sx_{n+1}$, then $\{x_{2n}\}$ converges to the coincidence quasi-best proximity point of $(T; S)$.

Proof. For $x_0 \in A$ define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. First, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2l}, Sx_{2n+1}) < \text{dist}(S(A), S(B)) + \varepsilon, \quad \forall l > n \geq N_0. \quad (*)$$

Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ satisfying

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \geq \text{dist}(S(A), S(B)) + \varepsilon_0$$

and

$$d(Sx_{2l_k-2}, Sx_{2n_k+1}) < \text{dist}(S(A), S(B)) + \varepsilon_0.$$

We have

$$\begin{aligned} \text{dist}(S(A), S(B)) + \varepsilon_0 &\leq d(Sx_{2l_k}, Sx_{2n_k+1}) \\ &\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + d(Sx_{2l_k-2}, Sx_{2n_k+1}) \\ &\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + \text{dist}(S(A), S(B)) + \varepsilon_0. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \rightarrow \text{dist}(S(A), S(B)) + \varepsilon_0.$$

Moreover, we have

$$\begin{aligned} \text{dist}(S(A), S(B)) + \varepsilon_0 &\leq d(Sx_{2l_k}, Sx_{2n_k+1}) = d(Tx_{2l_k-1}, Tx_{2n_k}) \\ &\leq \alpha d(Sx_{2l_k-1}, Sx_{2n_k}) + (1 - \alpha) \text{dist}(S(A), S(B)) \\ &= \alpha d(Tx_{2l_k-2}, Tx_{2n_k-1}) + (1 - \alpha) \text{dist}(S(A), S(B)) \\ &\leq \alpha d(Sx_{2l_k-2}, Sx_{2n_k-1}) + (1 - \alpha) \text{dist}(S(A), S(B)). \end{aligned}$$

Therefore, by letting $k \rightarrow \infty$ we obtain

$$\begin{aligned} \text{dist}(S(A), S(B)) + \varepsilon_0 &\leq \alpha (\text{dist}(S(A), S(B)) + \varepsilon_0) + (1 - \alpha) \text{dist}(S(A), S(B)) \\ &\leq \text{dist}(S(A), S(B)) + \varepsilon_0. \end{aligned}$$

This implies that $\alpha = 1$, which is a contradiction. That is, (*) holds. Now, assume $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ such that

$$d(Sx_{2l_k}, Sx_{n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(S(A), S(B))$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - \text{dist}(S(A), S(B)), \frac{\text{dist}(S(A), S(B))\alpha(\gamma)}{1 - \alpha(\gamma)} \right\}.$$

Let $N \in \mathbb{N}$ be such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(S(A), S(B)) + \varepsilon, \quad \forall n_k \geq N$$

and

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \leq \text{dist}(S(A), S(B)) + \varepsilon, \quad \forall l_k > n_k \geq N.$$

Uniform convexity of X implies that

$$\begin{aligned} \text{dist}(S(A), S(B)) &\leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2l_k}, \frac{1}{2})) \\ &\leq (\text{dist}(S(A), S(B)) + \varepsilon)(1 - \alpha(\gamma)) < \text{dist}(S(A), S(B)), \end{aligned}$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. By the fact that S is relatively anti-Lipschitzian on $A \cup B$, we have

$$d(x_{2l}, x_{2n}) \leq cd(Sx_{2l}, Sx_{2n}) \rightarrow 0, \quad l, n \rightarrow \infty,$$

that is, $\{x_{2n}\}$ is a Cauchy sequence. Since A is complete, there exists $p \in A$ such that $x_{2n} \rightarrow p$. Now, the result follows from a similar argument as in Theorem 2.8. \square

3. QUASI-CYCLIC-NONCYCLIC RELATIVELY CONTRACTION MAPPINGS

In this section, we introduce the class of quasi-cyclic-noncyclic relatively contraction mappings that contains the class of cyclic-noncyclic contraction mappings as a subclass. Next, we study the existence and convergence of coincidence best proximity points in the setting of convex metric spaces for quasi-cyclic-noncyclic relatively contraction mappings.

Definition 3.1. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \rightarrow X$ be two mappings. The pair $(T; S)$ is called a quasi-cyclic-noncyclic relatively contraction pair if it satisfies the following conditions:

- (1) $(T; S)$ is a quasi-cyclic-noncyclic pair on X ; that is,

$$T(A) \subseteq S(B), T(B) \subseteq S(A).$$

- (2) For some $\alpha \in (0, 1)$ and for each $(x, y) \in A \times B$ we have

$$d(Tx, Ty) \leq \alpha d(Sx, Sy) + (1 - \alpha) \text{dist}(A, B).$$

Note that in the above definition we do not have the inequality

$$\text{dist}(A, B) \leq d(Sx, Sy),$$

that is,

$$d(Tx, Ty) \leq d(Sx, Sy)$$

is not always true.

We emphasize that if $S = I$ or if $S(A) = A$ and $S(B) = B$, then the above definition reduces to Definition 1.3.

EXAMPLE 3.2. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -3]$ and $B = [3, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} -(x+1), & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} 3x+5, & \text{if } x \in A \\ 3x-7, & \text{if } x \in B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{3}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{3}(3y - 3x - 12) + \frac{2}{3}(6) \\ &= \alpha |Sx - Sy| + (1 - \alpha) \text{dist}(A, B). \end{aligned}$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$.

Lemma 3.3. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then we have*

$$d(Sx_{2n}, Sx_{2n+1}) \rightarrow \text{dist}(A, B).$$

Proof. We note that

$$\begin{aligned} \text{dist}(A, B) &\leq \text{dist}(S(A), S(B)) \leq d(Sx_{2n+1}, Sx_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}) \\ &\leq \alpha d(Sx_{2n}, Sx_{2n+1}) + (1 - \alpha) \text{dist}(A, B) \\ &= \alpha d(Tx_{2n-1}, Tx_{2n}) + (1 - \alpha) \text{dist}(A, B) \\ &\leq \alpha [\alpha d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha) \text{dist}(A, B)] \\ &\quad + (1 - \alpha) \text{dist}(A, B) \\ &= \alpha^2 d(Sx_{2n-1}, Sx_{2n}) + (1 - \alpha^2) \text{dist}(A, B) \\ &= \alpha^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1 - \alpha^2) \text{dist}(A, B) \\ &\leq \dots \\ &\leq \alpha^{2n} d(Tx_0, Tx_1) + (1 - \alpha^2) \text{dist}(A, B). \end{aligned}$$

Now, if $n \rightarrow \infty$, we conclude that

$$d(Sx_{2n}, Sx_{2n+1}) \rightarrow \text{dist}(A, B).$$

□

Remark 3.4. If the pair $(T; S)$ is a QCN relatively contraction pair such that

$$S(A) \subseteq A \text{ and } S(B) \subseteq B,$$

then we have

$$\text{dist}(A, B) \leq \text{dist}(S(A), S(B)).$$

Thus, by this assumption, the Lemma holds true.

Theorem 3.5. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. Assume S is continuous on A . For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A , then the pair $(T; S)$ has a coincidence best proximity point in A .*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \rightarrow p \in A$. we have

$$\begin{aligned} \text{dist}(A, B) &\leq \text{dist}(S(A), S(B)) \leq d(Tx_{2n_k-1}, Tp) \leq d(Sx_{2n_k-1}, Sp) \\ &\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}). \end{aligned}$$

By Lemma 3.3, if $k \rightarrow \infty$, we obtain that

$$d(Tx_{2n_k-1}, Tp) \rightarrow \text{dist}(A, B).$$

Moreover,

$$\begin{aligned} \text{dist}(A, B) &\leq \text{dist}(S(A), S(B)) \leq d(Sp, Tp) \\ &\leq d(Sp, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tp) \\ &= d(Sp, Sx_{2n_k}) + d(Tx_{2n_k-1}, Tp) \\ &\rightarrow \text{dist}(A, B), \end{aligned}$$

that is,

$$d(Sp, Tp) = \text{dist}(A, B).$$

□

Lemma 3.6. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) . Suppose $(T; S)$ is a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then $\{Sx_{2n}\}$, and $\{Sx_{2n+1}\}$ are bounded sequences in $S(A)$ and $S(B)$ respectively.*

Proof. Since

$$d(Sx_{2n}, Sx_{2n+1}) \rightarrow \text{dist}(A, B),$$

it suffices to verify that $\{Sx_{2n}\}$ is bounded in $S(A)$. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_2, Sx_{2N_0+1}) > M, \quad d(Sx_2, Sx_{2N_0-1}) \leq M,$$

where,

$$M > \max \left\{ \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \text{dist}(A, B), d(Sx_1, Sx_0) \right\}.$$

By the above assumption, we have

$$\begin{aligned} \frac{M - \text{dist}(A, B)}{\alpha^2} + \text{dist}(A, B) &< \frac{d(Sx_2, Sx_{2N_0+1}) - \text{dist}(A, B)}{\alpha^2} + \text{dist}(A, B) \\ &\leq \frac{d(Sx_2, Sx_{2N_0+1}) + (\alpha^2 - 1)d(Sx_2, Sx_{2N_0+1})}{\alpha^2} \\ &= d(Sx_2, Sx_{2N_0+1}) = d(Tx_1, Tx_{2N_0}) \\ &\leq d(Sx_1, Sx_{2N_0}) = d(Tx_0, Tx_{2N_0-1}) \\ &= d(Sx_0, Sx_{2N_0-1}) \\ &\leq d(Sx_0, Sx_2) + d(Sx_2, Sx_{2N_0-1}) \\ &\leq d(Sx_0, Sx_2) + M. \end{aligned}$$

This implies that

$$\frac{M - \text{dist}(A, B)}{\alpha^2} + \text{dist}(A, B) < d(Sx_0, Sx_2) + M,$$

or,

$$M - (1 - \alpha^2)\text{dist}(A, B) < \alpha^2[d(Sx_0, Sx_2) + M].$$

and finally,

$$(1 - \alpha^2)M < \alpha^2 d(Sx_0, Sx_2) + (1 - \alpha^2)\text{dist}(A, B).$$

Now, we conclude that

$$M < \frac{\alpha^2}{1 - \alpha^2} d(Sx_0, Sx_2) + \text{dist}(A, B),$$

which is a contradiction by the choice of M . \square

Theorem 3.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that $S(A)$ is boundedly compact. Suppose $(T; S)$ is a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. If S is relatively anti-Lipschitzian and continuous on A , then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(A, B).$$

Proof. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. According to Lemma 3.6, $\{Sx_{2n}\}$ is bounded in $S(A)$, on the other hand $S(A)$ is boundedly compact, so that there exists a subsequence $\{Sx_{2n_k}\}$ of $\{Sx_{2n}\}$ such that

$$Sx_{2n_k} \rightarrow Sp,$$

for some $p \in A$. We know that S is relatively anti-Lipschitzian, therefore

$$d(x_{2n_k}, p) \leq cd(Sx_{2n_k}, Sp) \rightarrow 0, \quad k \rightarrow \infty.$$

This implies that $\{x_{2n_k}\}$ is a convergent subsequence of $\{x_{2n}\}$, hence the result follows from Theorem 3.5. \square

In the following we give examples to show that there exists a coincidence best proximity point that is not a fixed point for S .

EXAMPLE 3.8. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -3]$ and $B = [3, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} 3 - x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} 2x + 6, & \text{if } x \in A \\ 2x, & \text{if } x \in B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{2}(2y - 2x - 6) + \frac{1}{2}(6) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(A, B). \end{aligned}$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T; S)$ follows from Theorem 3.7; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(A, B) = 0 \text{ or } 3 - p - 2p - 6 = 6,$$

which implies that $p = -3$.

EXAMPLE 3.9. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -4]$ and $B = [4, +\infty)$ define $T, S : X \rightarrow X$ by

$$Tx := \begin{cases} 4 - x, & \text{if } x \in A \cup B \\ 0, & \text{ow.} \end{cases} \quad \text{and} \quad Sx := \begin{cases} 4x + 16, & \text{if } x \in A \\ 4x - 8, & \text{if } x \in B \\ 0, & \text{ow.} \end{cases}$$

Then $(T; S)$ is a QCN relatively contraction pair with $\alpha = \frac{1}{4}$. Indeed, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| &= (y - x) \leq \frac{1}{4}(4y - 4x - 24) + \frac{3}{4}(8) \\ &= \alpha|Sx - Sy| + (1 - \alpha)\text{dist}(A, B). \end{aligned}$$

Also, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Finally, the existence of coincidence best proximity point of the pair $(T; S)$ follows from Theorem 3.7; that is, there exists $p \in A$ such that

$$|Tp - Sp| = \text{dist}(A, B) = 8 \text{ or } 4 - p - 4p - 16 = 8,$$

which implies that $p = -4$.

Lemma 3.10. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space (X, d, \mathcal{W}) such that $S(A)$ is convex. Suppose $(T; S)$ is a QCN relatively contraction pair defined on X and $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. For $x_0 \in A$, define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \rightarrow 0, \quad d(Sx_{2n+3}, Sx_{2n+1}) \rightarrow 0.$$

Proof. We prove that $d(Sx_{2n+2}, Sx_{2n}) \rightarrow 0$. Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exists $n_k \geq k$ such that

$$d(Sx_{2n_k+2}, Sx_{2n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - \text{dist}(A, B), \frac{\text{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)} \right\}.$$

By Lemma 3.3, we know that $d(Sx_{2n_k}, Sx_{2n_k+1}) \rightarrow \text{dist}(A, B)$, so there exists $N \in \mathbb{N}$ such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon,$$

$$d(Sx_{2n_k+2}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon$$

and

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \geq \varepsilon_0 > \gamma(\text{dist}(A, B) + \varepsilon).$$

It now follows from the uniformly convexity of X and the convexity of $S(A)$ that

$$\begin{aligned} \text{dist}(A, B) &\leq \text{dist}(S(A), S(B)) \leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \frac{1}{2})) \\ &\leq (\text{dist}(A, B) + \varepsilon)(1 - \alpha(\gamma)) \\ &< \text{dist}(A, B) + \frac{\text{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)}(1 - \alpha(\gamma)) \\ &= \text{dist}(A, B), \end{aligned}$$

which is a contradiction. Similarly, we see that $d(Sx_{2n+3}, Sx_{2n+1}) \rightarrow 0$. \square

The following Theorem guarantees the existence and convergence of coincidence best proximity points for QCN relatively contraction mappings in the setting of uniformly convex metric spaces.

Theorem 3.11. *Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d; \mathcal{W})$ such that $S(A)$ is convex. Suppose $(T; S)$ is a QCN relatively contraction pair defined on X such that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Assume that $\text{dist}(A, B) \leq \text{dist}(S(A), S(B))$. Then there exists $p \in A$ such that*

$$d(Sp, Tp) = \text{dist}(A, B).$$

Further, if $x_0 \in A$ and $Tx_n = Sx_{n+1}$, then $\{x_{2n}\}$ converges to the coincidence best proximity point of $(T; S)$.

Proof. For $x_0 \in A$ define $Tx_n = Sx_{n+1}$ for each $n \geq 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. First, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2l}, Sx_{2n+1}) < \text{dist}(A, B) + \varepsilon, \quad \forall l > n \geq N_0. \quad (*)$$

Assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ satisfying

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \geq \text{dist}(A, B) + \varepsilon_0, \quad d(Sx_{2l_k-2}, Sx_{2n_k+1}) < \text{dist}(A, B) + \varepsilon_0.$$

Note that

$$\begin{aligned} \text{dist}(A, B) + \varepsilon_0 &\leq d(Sx_{2l_k}, Sx_{2n_k+1}) \\ &\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + d(Sx_{2l_k-2}, Sx_{2n_k+1}) \\ &\leq d(Sx_{2l_k}, Sx_{2l_k-2}) + \text{dist}(A, B) + \varepsilon_0. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \rightarrow \text{dist}(A, B) + \varepsilon_0.$$

Moreover, we have

$$\begin{aligned} \text{dist}(A, B) + \varepsilon_0 &\leq d(Sx_{2l_k}, Sx_{2n_k+1}) = d(Tx_{2l_k-1}, Tx_{2n_k}) \\ &\leq \alpha d(Sx_{2l_k-1}, Sx_{2n_k}) + (1 - \alpha)\text{dist}(A, B) \\ &= \alpha d(Tx_{2l_k-2}, Tx_{2n_k-1}) + (1 - \alpha)\text{dist}(A, B) \\ &\leq \alpha d(Sx_{2l_k-2}, Sx_{2n_k-1}) + (1 - \alpha)\text{dist}(A, B). \end{aligned}$$

Therefore, by letting $k \rightarrow \infty$ we obtain

$$\text{dist}(A, B) + \varepsilon_0 \leq \alpha(\text{dist}(A, B) + \varepsilon_0) + (1 - \alpha)\text{dist}(A, B) \leq \text{dist}(A, B) + \varepsilon_0.$$

This implies that $\alpha = 1$, which is a contradiction. That is, (*) holds. Now, assume that $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $l_k > n_k \geq k$ such that

$$d(Sx_{2l_k}, Sx_{n_k}) \geq \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \text{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - \text{dist}(A, B), \frac{\text{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)} \right\}.$$

Let $N \in \mathbb{N}$ be such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon, \quad \forall n_k \geq N$$

and

$$d(Sx_{2l_k}, Sx_{2n_k+1}) \leq \text{dist}(A, B) + \varepsilon, \quad \forall l_k > n_k \geq N.$$

Uniformly convexity of X implies that

$$\begin{aligned} \text{dist}(A, B) &\leq \text{dist}(S(A), S(B)) \leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2l_k}, \frac{1}{2})) \\ &\leq (\text{dist}(A, B) + \varepsilon)(1 - \alpha(\gamma)) < \text{dist}(A, B), \end{aligned}$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. By the fact that S is relatively anti-Lipschitzian on $A \cup B$, we have

$$d(x_{2l}, x_{2n}) \leq cd(Sx_{2l}, Sx_{2n}) \rightarrow 0, \quad l, n \rightarrow \infty,$$

that is, $\{x_{2n}\}$ is Cauchy. Since A is complete, there exists $p \in A$ such that $x_{2n} \rightarrow p$. Now, the result follows from a similar argument as in the proof of Theorem 3.5. \square

ACKNOWLEDGMENTS

The authors would like to thank the referee for useful and helpful comments and suggestions.

REFERENCES

1. A. Abkar, M. Gabeleh, Best Proximity Points for Cyclic Mappings in Ordered Metric Spaces, *J. Optim. Theory Appl.*, **150**, (2011), 188–193.
2. M. A. Al-Thagafi, N. Shahzad, Convergence and Existence Results for Best Proximity Points, *Nonlinear Anal.*, **70**, (2009), 3665–3671.
3. M. Borcut, V. Berinde, Tripled Fixed Point Theorems for Contractive Type Mappings in Partially Ordered Metric Spaces, *Nonlinear Anal.*, **74**, (2011), 4889–4897.
4. Y. J. Cho, A. Gupta, E. Karapinar, P. Kumam, W. Sintunawarat, Tripled Best Proximity Point Theorem in Metric Spaces, *Math. Ineq. Appl.*, **16**, (2013), 1197–1216.
5. M. De la Sen, Some Results on Fixed and Best Proximity Points of Multivalued Cyclic Self Mappings with a Partial Order, *Abst. Appl. Anal.*, **2013**, (2013), Article ID 968492, 11 pages.
6. M. De la Sen, R. P. Agarwal, Some Fixed Point-Type Results for a Class of Extended Cyclic Self Mappings with a More General Contractive Condition, *Fixed Point Theory Appl.*, **59**, (2011), 14 pages.
7. C. Di Bari, T. Suzuki, C. Verto, Best Proximity Points for Cyclic Meir-Keeler Contractions, *Nonlinear Anal.*, **69**, (2008), 3790–3794.
8. A. A. Eldred, P. Veeramani, Existence and Convergence of Best Proximity Points, *J. Math. Anal. Appl.*, **323**, (2006), 1001–1006.
9. A. A. Eldred, W. A. Kirk, P. Veeramani, Proximal Normal Structure and Relatively Nonexpansive Mappings, *Studia Math.*, **171**, (2005), 283–293.
10. R. Espinola, M. Gabeleh, P. Veeramani, On the Structure of Minimal Sets of Relatively Nonexpansive Mappings, *Numer. Funct. Anal. Optim.*, **34**, (2013), 845–860.
11. A. F. Leon, M. Gabeleh, Best Proximity Pair Theorems for Noncyclic Mappings in Banach and Metric Spaces, *Fixed Point Theory*, **17**, (2016), 63–84.
12. H. Fukhar-ud-din, A. R. Khan, Z. Akhtar, Fixed Point Results for a Generalized Non-expansive Map in Uniformly Convex Metric Spaces, *Nonlinear Anal.*, **75**, (2012), 4747–4760.
13. M. Gabeleh, H. Lakzian, N. Shahzad, Best Proximity Points for Asymptotic Pointwise Contractions, *J. Nonlinear Convex Anal.*, **16**, (2015), 83–93.
14. M. Gabeleh, O. Olela Otafudu, N. Shahzad, Coincidence Best Proximity Points in Convex Metric Spaces, *Filomat*, **32**, (2018), 1–12.
15. J. Garcia Falset, O. Mlesinte, Coincidence Problems for Generalized Contractions, *Applicable Anal. Discrete Math.*, **8**, (2014), 1–15.
16. N. Hussain, A. Latif, P. Salimi, Best Proximity Point Results in G -Metric Spaces, *Abst. Appl. Anal.*, (2014), Article ID 837943.
17. E. Karapinar, Best Proximity Points of Kannan Type Cyclic Weak ϕ -Contractions in Ordered Metric Spaces, *An. St. Univ. Ovidius Constanta.*, **20**, (2012), 51–64.
18. W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed Points for Mappings Satisfying Cyclic Contractive Conditions, *Fixed point Theory*, **4**, (2003), 79–86.
19. R. Lashkaripour, J. Hamzehnejadi, Generalization of the Best Proximity Point, *J. Inequalities And Special Functions.*, **4**, (2017), 136–147.
20. Z. Mustafa, *A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory [Ph.D. Thesis]*, The University of Newcastle, New South Wales, Australia., 2005.
21. Z. Mustafa, H. Obiedat, F. Awawdeh, Some Fixed Point Theorem for Mapping on Complete G -Metric Spaces, *Fixed Point Theory Appl.*, (2008), Article ID 189870.
22. Z. Mustafa, B. Sims, A New Approach to Generalized Metric Spaces, *J. Nonlinear Convex Anal.*, (2006), 289–297.

23. V. Pragadeeswarar, M. Marudai, Best Proximity Points: Approximation and Optimization in Partially Ordered Metric Spaces, *Optim. Lett.*, **7**, (2013), 1883–1892.
24. T. Shimizu, W. Takahashi, Fixed Points of Multivalued Mappings in Certain Convex Metric Spaces, *Topological Methods in Nonlin. Anal.*, **8**, (1996), 197–203.
25. T. Suzuki, M. Kikkawa, C. Vetro, The Existence of Best Proximity Points in Metric Spaces with to Property UC, *Nonlinear Anal.*, **71**, (2009), 2918–2926.
26. W. Takahashi, A Convexity in Metric Space and Nonexpansive Mappings, *Kodai Math. Sem. Rep.*, **22**, (1970), 142–149.
27. T. Van An, N. V. An, V. T. Le Hang, A New Approach to Fixed Point Theorems on G -Metric Spaces, *Topology and its Applications.*, **160**, (2013), 1486–1493.