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On Eulerianity and Hamiltonicity in Annihilating-ideal Graphs

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Abstract. Let R be a commutative ring with identity, and $\mathrm{A}(R)$ be the set of ideals with non-zero annihilator. The annihilating-ideal graph of R is defined as the graph $AG(R)$ with the vertex set $\mathrm{A}(R)^* = \mathrm{A}(R)\setminus\{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. In this paper, conditions under which $AG(R)$ is either Eulerian or Hamiltonian are given.

Keywords: Annihilating-ideal graph, Eulerian graphs, Hamiltonian graphs.

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1. Introduction

Assigning a graph to an algebraic structure make a bridge between two different worlds of mathematics. Usually study of such graphs lead to arising interesting algebraic and combinatorics problems and one may find a new perception of algebraic structures. Therefore, the study of graphs associated with rings has attracted many researchers in recent years. There are a lot of papers in this field; for instance see [2, 3, 5, 8].

Throughout this paper, all rings are assumed to be commutative with identity. The set of all ideals of a ring R is denoted by $\mathrm{I}(R)$. For a subset T of

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a ring R we let $T^* = T \setminus \{0\}$. Furthermore, if I is an ideal of R, by $Ann(I)$, we mean the annihilator of I. An ideal I of R is called an *annihilating-ideal* if there exists a non-zero ideal J such that $IJ = 0$. The notation $A(R)$ is used to denote the set of all annihilating-ideals of R. The *socle* of a ring R, denoted by $soc(R)$, is the sum of all minimal ideals of R. If there are no minimal ideals, this sum is defined to be zero. A ring R is said to be *local* if it has a unique maximal ideal. A local ring R with maximal ideal $\mathfrak{m}$ is presented by $(R, \mathfrak{m})$. Let $(R, \mathfrak{m})$ be a local ring. The *associated degree of* R is n if it is the smallest positive integer such that $\mathfrak{m}^n = 0$. If $\mathfrak{m}^n \neq 0$, for every $n \geq 1$, then we say that the associated degree of R is ∞. For any undefined notation or terminology in ring theory, we refer the reader to [4].

We now recall some basic graph theoretic facts: Let G be a graph with the vertex set $V(G)$. The size of G is denoted by $e(G)$. We write $u - v$, to denote an edge with ends u, v. Also, a complete graph of order n is denoted by K_n. For any $x \in V(G)$, $N_G(x)$ represents the set of neighbors of x in G and $|N_G(x)| = deg_G(x)$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1), v \in V(G_2)\}$. An *Eulerian circuit* in an undirected graph is a circuit that uses each edge exactly once. If such a circuit exists, then the graph is called *Eulerian*. It is well-known that a connected graph G is Eulerian if and only if every vertex of G has even degree. Similarly, a *Hamiltonian cycle* is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*. For any undefined notation or terminology in graph theory, we refer the reader to [9].

Let R be a ring. The *annihilating-ideal graph* of R, denoted by $AG(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. The annihilating-ideal graph was first introduced in [6], and some of the properties of the annihilating-ideal graph have been studied. Further results on annihilating-ideal graphs may be found in [1, 7] and [10]. This paper is devoted to study Eulerian and Hamiltonian annihilating-ideal graphs.

2. Eulerianity of $AG(R)$

Throughout this section $R = R_1 \times R_2$ is a decomposable ring, $|\mathrm{I}(R_1)| = t_1$ and $|\mathrm{I}(R_2)| = t_2$, where t_1, t_2 are two positive integers. The main goal of this section is study the relations between the Eulerianity of $AG(R)$ and Eulerianity of $AG(R_i)$, for $i = 1, 2$. First we need to compute the size of $AG(R)$.

Theorem 2.1. *Let R be a ring. Then* $e(AG(R)) = 2(x + a)(y + b) + (m + 1)x + (1 + n)y - ab + bn + am$, *where* $a = t_1 - 1$, $b = t_2 - 1$, $e(AG(R_1)) = x$,

$e(AG(R_2)) = y$, n is the number of non-trivial ideals I of R_1 with $I^2 = 0$ and m is the number of non-trivial ideals J of R_2 with $J^2 = 0$.

Proof. Suppose that (I, J) is a vertex of $AG(R)$, where $I \in I(R_1)$ and $J \in I(R_2)$. To prove the theorem the following cases are considered:

Case 1. Let $I = R_1$ and $J \neq 0, R_2$.

(i) If $J^2 \neq 0$, then $deg_{AG(R)}(I, J) = deg_{AG(R_2)}(J)$, as

$$N_{AG(R)}((I, J)) = \{(0, J') \mid J' \in N_{AG(R_2)}(J)\}.$$

(ii) If $J^2 = 0$, then $deg_{AG(R)}(I, J) = deg_{AG(R_2)}(J) + 1$, since

$$N_{AG(R)}((I, J)) = \{(0, J') \mid J' \in N_{AG(R_2)}(J)\} \cup \{(0_{R_1}, J)\}.$$

Case 2. Let $I \neq 0, R_1$ and $J = R_2$.

(i) If $I^2 \neq 0$, then $deg_{AG(R)}(I, J) = deg_{AG(R_1)}(I)$.

(ii) If $I^2 = 0$, then $deg_{AG(R)}(I, J) = deg_{AG(R_1)}(I) + 1$.

Case 3. Let $I \neq 0, R_1$ and $J \neq 0, R_2$.

(i) If $I^2 \neq 0$ and $J^2 \neq 0$, then

$$deg_{AG(R)}(I, J) = (deg_{AG(R_1)}(I) + 1)(deg_{AG(R_2)}(J) + 1) - 1,$$

as

$$N_{AG(R)}((I, J)) = \{(I', J') \mid I' \in \{0_{R_1}\} \cup N_{AG(R_1)}(I), J' \in \{0_{R_2}\} \cup N_{AG(R_2)}(J)\} \setminus \{(0_{R_1}, 0_{R_2})\}.$$

(ii) If $I^2 = 0$ and $J^2 \neq 0$, then

$$deg_{AG(R)}(I, J) = (deg_{AG(R_1)}(I) + 2)(deg_{AG(R_2)}(J) + 1) - 1,$$

since

$$N_{AG(R)}((I, J)) = \{(I', J') \mid I' \in \{0_{R_1}, I\} \cup N_{AG(R_1)}(I), J' \in \{0_{R_2}\} \cup N_{AG(R_2)}(J)\} \setminus \{(0_{R_1}, 0_{R_2})\}.$$

(iii) If $I^2 \neq 0$ and $J^2 = 0$, then

$$deg_{AG(R)}(I, J) = (deg_{AG(R_1)}(I) + 1)(deg_{AG(R_2)}(J) + 2) - 1.$$

(iv) If $I^2 = 0$ and $J^2 = 0$, then

$$deg_{AG(R)}(I, J) = (deg_{AG(R_1)}(I) + 2)(deg_{AG(R_2)}(J) + 2) - 2.$$

Case 4. Let $I = 0$ and $J \neq 0, R_2$.

(i) If $J^2 \neq 0$, then

$$deg_{AG(R)}(I, J) = t_1(deg_{AG(R_2)}(J) + 1) - 1 = t_1 deg_{AG(R_2)}(J) + t_1 - 1,$$

since

$$N_{AG(R)}((I, J)) = \{(I', J') \mid I' \in I(R_1), J' \in \{0_{R_2}\} \cup N_{AG(R_2)}(J)\} \setminus \{(0_{R_1}, 0_{R_2})\}.$$

(ii) If $J^2 = 0$, then

$$deg_{AG(R)}(I, J) = t_1(deg_{AG(R_2)}(J) + 2) - 2,$$

as

$$N_{AG(R)}((I, J)) = \{(I', J') \mid I' \in I(R_1), J' \in \{0_{R_2}, J\} \cup N_{AG(R_2)}(J)\} \setminus \{(0_{R_1}, 0_{R_2}), (0_{R_1}, J)\}.$$

Case 5. Let $I \neq 0, R_1$ and $J = 0$.

(i) If $I^2 \neq 0$, then $deg_{AG(R)}(I, J) = t_2(deg_{AG(R_1)}(I)+1)-1 = t_2 deg_{AG(R_1)}(I)+ t_2 - 1$.

(ii) If $I^2 = 0$, then $deg_{AG(R)}(I, J) = t_2(deg_{AG(R_1)}(I) + 2) - 2$.

Case 6. Let $I = 0$ and $J = R_2$. Then

$$deg_{AG(R)}(I, J) = t_1 - 1,$$

since $N_{AG(R)}((I, J)) = \{(I', 0_{R_2}) \mid I' \in I(R_1)\} \setminus \{(0_{R_1}, 0_{R_2})\}$.

Case 7. Let $I = R_1$ and $J = 0$. Then $deg_{AG(R)}(I, J) = t_2 - 1$.

Since the sum of the degrees of the vertices in the graph $AG(R)$ is twice the number of edges of $AG(R)$, the total number of edges of $AG(R)$ is

$$e(AG(R)) = 2(x + a)(y + b) + (m + 1)x + (1 + n)y - ab + bn + am,$$

where $a = t_1 - 1$, $b = t_2 - 1$, $e(AG(R_1)) = x$, $e(AG(R_2)) = y$, n is the number of non-trivial ideals I of R_1 with $I^2 = 0$ and m is the number of non-trivial ideals J of R_2 with $J^2 = 0$. □

In light of Theorem 2.1, we have the following corollary.

Corollary 2.2. *The graph $AG(\mathbb{Z}_n)$ is not Eulerian, for every positive integer n.*

Proof. If $n \leq 5$, then there is nothing to prove. Hence assume that $n \geq 6$. If $(\mathbb{Z}_n, \mathfrak{m})$ is a local ring of associated degree $l \geq 3$, then $deg(\mathfrak{m}) = 1$, as $N(\mathfrak{m}) = \{\mathfrak{m}^{l-1}\}$. If $\mathfrak{m}^2 = 0$, clearly $n = p^2$, for some prime number p. So $AG(\mathbb{Z}_n) = K_1$. Thus $AG(\mathbb{Z}_n)$ is not Eulerian. Now, suppose that $\mathbb{Z}_n$ is not local. If $\mathbb{Z}_n = F_1 \times \cdots \times F_k$, where every F_i is a field, then $deg(F_1 \times \cdots \times F_{k-1} \times 0) = 1$ and so $AG(\mathbb{Z}_n)$ is not Eulerian. Therefore, we may assume that $\mathbb{Z}_n = R_1 \times R_2$, where $(R_1, \mathfrak{m})$ is local which is not a field. To complete the proof, it is enough to show that $deg(\mathfrak{m}, R_2) = 1$. If $\mathfrak{m}^2 \neq 0$, then by (i) of *Case* 2 in the proof of Theorem 2.1, $deg(\mathfrak{m}, R_2) = deg(\mathfrak{m}) = 1$. If $\mathfrak{m}^2 = 0$, then by (ii) of *Case* 2 in the proof of Theorem 2.1, $deg(\mathfrak{m}, R_2) = deg(\mathfrak{m}) + 1 = 1$, as desired. □

To prove Theorem 2.7, the following lemmas are needed.

Lemma 2.3. *Suppose that $I \in I(R_1)$ and $I^2 = 0$. If $AG(R)$ is Eulerian, then $AG(R_1)$ is not Eulerian.*

Proof. Suppose that $AG(R)$ is Eulerian. Then $deg_{AG(R)}(I, R_2) = deg_{AG(R_1)}(I) + 1$ is even (by (ii) of $Case\,2$ in the proof of Theorem 2.1). Thus $deg_{AG(R_1)}(I)$ is odd and hence $AG(R_1)$ is not Eulerian. □

The converse of Lemma 2.3 is not true in general, see the following example.

Example 2.4. Let $R_1 = \mathbb{Z}_{16}$. Obviously, $AG(R_1)$ is not Eulerian ($AG(R_1) = K_{1,2}$). Moreover, if we let $R = \mathbb{Z}_{16} \times \mathbb{Z}_3$, then $AG(R)$ is not Eulerian too, as $deg_{AG(R)}(\mathbb{Z}_{16}, 0_{R_2}) = 1$.

By a similar argument, one may prove the following lemma.

Lemma 2.5. *Suppose that $J \in I(R_2)$ and $J^2 = 0$. If $AG(R)$ is Eulerian, then $AG(R_2)$ is not Eulerian.*

Remark 2.6. Let (I, J) be a vertex of $AG(R)$, where $I \in I(R_1)$ and $J \in I(R_2)$. By Lemmas 2.3 and 2.5, if either $I^2 = 0$ or $J^2 = 0$, then the Eulerianity of $AG(R)$ does not imply the Eulerianity of $AG(R_i)$, for $i = 1, 2$. Thus we suppose that $I^2 \neq 0$ and $J^2 \neq 0$. It is worthy to mention that if $I^2 \neq 0$ and $J^2 \neq 0$, then by Theorem 2.1, the degrees of the vertices are in the following forms:
(1) $t_1 deg_{AG(R_2)}(J) + t_1 - 1$ and $t_2 deg_{AG(R_1)}(I) + t_2 - 1$.
(2) $t_1 - 1$ and $t_2 - 1$.
(3) $deg_{AG(R_2)}(J)$ and $deg_{AG(R_1)}(I)$.
(4) $(deg_{AG(R_1)}(I) + 1)(deg_{AG(R_2)}(J) + 1) - 1$.

Theorem 2.7. *Let $AG(R)$ be Eulerian and $I^2 \neq 0$, $J^2 \neq 0$, for every vertex (I, J) of $AG(R)$. Then both of $AG(R_1)$ and $AG(R_2)$ are Eulerian.*

Proof. Let $AG(R)$ be Eulerian. Then the degree of every vertex of $AG(R)$ is even and so by form (3) in Remark 2.6, $deg_{AG(R_1)}(I)$ and $deg_{AG(R_2)}(J)$ are even (for non-trivial ideals I of R_1 and J of R_2). Hence both of $AG(R_1)$ and $AG(R_2)$ are Eulerian. □

The converse of Theorem 2.7 is not true in general. For example, suppose that t_1 and t_2 are even. If $AG(R_1)$ and $AG(R_2)$ are Eulerian, then forms (3) and (4) in Remark 2.6 are even but forms (1) and (2) are odd and hence $AG(R)$ is not Eulerian.

The last result of this section gives a condition under which the Eulerianity of $AG(R_1)$ and $AG(R_2)$ implies the Eulerianity of $AG(R)$.

Theorem 2.8. *Let $AG(R_i)$ be Eulerian, for $i = 1, 2$. Then $AG(R)$ is Eulerian if and only if t_1 and t_2 are odd.*

Proof. First suppose that t_1 and t_2 are odd and (I, J) is a vertex of $AG(R)$. Since $AG(R_1)$ and $AG(R_2)$ are Eulerian, we deduce that $deg_{AG(R_1)}(I)$ and $deg_{AG(R_2)}(J)$ are even and hence forms (3) and (4) in Remark 2.6 are even. By the hypothesis, t_1 and t_2 are odd and so a simple check shows that forms (1) and (2) are even. Therefore, $AG(R)$ is Eulerian.

Conversely, suppose that $AG(R)$ is Eulerian. Then form (2) is even, i.e., t_1 and t_2 are odd. □

3. Hamiltonicity of $AG(R)$

In this section Hamiltonian annihilating-ideal graphs are investigated. It is assumed that, in this section, R is Artinian and hence every ideal of R is annihilating. First it is shown that annihilating-ideal graph of a decomposable ring is not Hamiltonian.

Theorem 3.1. *If R is a decomposable ring, then $AG(R)$ is not Hamiltonian.*

Proof. Let $R = R_1 \times R_2$, where R_1 and R_2 are two rings. We consider the following cases:

Case 1. Suppose $|I(R_1)| = |I(R_2)| = 2$. Then $AG(R) = K_2$ and so it is not Hamiltonian.

Case 2. Suppose $|I(R_1)| \geqslant 3$ and $|I(R_2)| = 2$. Then $N_{AG(R)}((R_1 \times 0)) = \{0 \times R_2\}$ and so $deg_{AG(R)}(R_1 \times 0) = 1$. Hence $AG(R)$ does not contain a Hamiltonian cycle.

Case 3. Suppose $|I(R_1)| \geqslant 3$ and $|I(R_2)| \geqslant 3$. Set

$$A = \{I \times R_2 \mid I \lhd R_1\}, B = \{J \times 0 \mid 0 \neq J \lhd R_1\}.$$

It is clear that $A \cup B \subseteq V(AG(R))$ and $|A| = |B|$. Suppose to the contrary, C is a Hamiltonian cycle of $AG(R)$. Thus every vertex of A is between two vertices of B (in C). Since $V(AG(R)) \setminus (A \cup B) \neq \emptyset$, we deduce that $|A| < |B|$, a contradiction. Hence $AG(R)$ is not Hamiltonian. □

By Theorem 3.1, to find Hamiltonian annihilating ideal graphs we have to study local rings.

Remark 3.2. Let $(R, \mathfrak{m})$ be a local ring of associated degree n. If $\mathfrak{m}$ is principal, then by [4, Proposition 8.8], $I(R) = \{0, R, \mathfrak{m}, \mathfrak{m}^2, \ldots, \mathfrak{m}^{n-1}\}$ and so $deg_{AG(R)}(\mathfrak{m}) = 1$, i.e., $AG(R)$ is not Hamiltonian. If we remove $\mathfrak{m}$ from $AG(R)$, then $AG(R) \setminus \{\mathfrak{m}\}$ contains exactly one Hamiltonian cycle; see the next result.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a local ring of associated degree n. Then the following statements are equivalent:*

(1) $\mathfrak{m}$ *is principal and* $|I(R)| \geq 6$.

(2) *If n is odd, then* $\mathfrak{m}^{n-1} - \mathfrak{m}^2 - \mathfrak{m}^{n-2} - \cdots - \mathfrak{m}^{[n/2]} - \mathfrak{m}^{n-[n/2]} - \mathfrak{m}^{n-1}$ *is the only Hamiltonian cycle of $AG(R) \setminus \{\mathfrak{m}\}$ and if n is even, then* $\mathfrak{m}^{n-1} - \mathfrak{m}^2 - \mathfrak{m}^{n-2} - \cdots - \mathfrak{m}^{[n/2]} - \mathfrak{m}^{n-1}$ *is the only Hamiltonian cycle of $AG(R) \setminus \{\mathfrak{m}\}$.*

(3) $I(R) = \{0, R, \mathfrak{m}, \mathfrak{m}^2, \ldots, \mathfrak{m}^{n-1}\}, n \geq 6$.

We note that the condition $|I(R)| \geq 6$ in Theorem 3.3 is necessary. If $|I(R)| \leq 5$, then it is not hard to check that $AG(R)$ is one of K_1, K_2 or $K_{1,2}$ and so it is not Hamiltonian.

Corollary 3.4. *The following statements hold:*

(1) *The graph $AG(\mathbb{Z}_n)$ is not Hamiltonian, for every positive integer n.*

(2) *Let m be a positive integer. Then $AG(\mathbb{Z}_{p^m}) \setminus \{\mathfrak{m}\}$ has a Hamiltonian cycle if and only if $m \geq 6$.*

Let R be a ring and I be an ideal of R. We denote by $A(I)$ the set of all ideals of R which are contained in I.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring. Then the following statements are equivalent:*

(1) *$AG(R) = K_n \vee \overline{K}_m$ for some non-negative integers n, m.*

(2) *For every ideal I of R, either $\mathrm{Ann}(I) \subseteq Soc(R) \cup I$ or $\mathrm{Ann}(I) = \mathfrak{m}$.*

Proof. (1) $\Longrightarrow$ (2) Let $AG(R) = K_n \vee \overline{K}_m$. If $m = 0$, then $AG(R)$ is a complete graph. Thus for every ideal of R say, I, $\mathrm{Ann}(I) \subseteq Soc(R)$ (note that in this case $Soc(R) = \mathfrak{m}$). So let $m \neq 0$ and I be a vertex of $AG(R)$. Since $AG(R) = K_n \vee \overline{K}_m$, $I \in V(K_n)$ or $I \in V(\overline{K}_m)$. If $I \in V(K_n)$, then I is adjacent to every other vertex and thus $IJ = (0)$ for every $I(R) \setminus \{I\}$. This implies that $\mathrm{Ann}(I) \cup I = \mathfrak{m}$. Hence $I \subseteq \mathrm{Ann}(I)$ or $\mathrm{Ann}(I) \subseteq I$. If $\mathrm{Ann}(I) \subseteq I$, then $I = \mathfrak{m}$ and hence $\mathfrak{m}$ is adjacent to every other vertex. This means that $\mathfrak{m} = \mathrm{Ann}(\mathfrak{m})$. Thus $AG(R)$ is a complete graph, a contradiction. So $I \subseteq \mathrm{Ann}(I)$ and hence $\mathrm{Ann}(I) = \mathfrak{m}$. If $I \in V(\overline{K}_m)$, then $\mathrm{Ann}(I) \in V(K_n) \cup \{I\}$. To complete the proof, we show that $V(K_n) = A(Soc(R))$. If $I \in A(Soc(R))$, then $\mathrm{Ann}(I) = \mathfrak{m}$ and thus I is adjacent to every other vertex. This implies that $I \in V(K_n)$ and hence $A(Soc(R)) \subseteq V(K_n)$. Now, let $I \in V(K_n)$. Since I is adjacent to every other vertex, we can easily get $\mathrm{Ann}(I) = \mathfrak{m}$ and so $I \in A(Soc(R))$.

(2) $\Longrightarrow$ (1) If we put $|A(Soc(R))| = n$ and $|A(R)^* \setminus A(Soc(R))| = m$, then $AG(R) = K_n \vee \overline{K}_m$. □

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a local ring and $AG(R) = K_n \vee \overline{K}_m$, for some non-negative integers n, m. Then $AG(R)$ is Hamiltonian if and only if $n \geq m$.*

We close this paper with the following example.

Example 3.7. (1) If $R = \dfrac{\mathbb{Z}_2[x, y]}{(x, y)^2}$, then $AG(R) = K_4 \vee \overline{K}_0$. Since $4 > 0$, $AG(R)$ is Hamiltonian.

(2) If $R = \mathbb{Z}_{16}$, then $AG(R) = K_1 \vee \overline{K}_2$. Since $2 > 1$, $AG(R)$ is not Hamiltonian.

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