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Some Weighted Integral Inequalities for Generalized Conformable Fractional Calculus

Hüseyin Budak*, Fuat Usta, Mehmet Zeki Sarikaya

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.

E-mail: hsyn.budak@gmail.com
E-mail: fuatusta@duzce.edu.tr
E-mail: sarikayamz@gmail.com

Abstract. In this paper, we have obtained weighted versions of Ostrowski, Čebysev and Grüss type inequalities for conformable fractional integrals which is given by Katugompola. By using the Katugampola definition for conformable calculus, the present study confirms previous findings and contributes additional evidence that provide the bounds for more general functions.

Keywords: Ostrowski inequality, Čebysev inequality, Grüss inequality, Conformable fractional integrals.

2010 Mathematics subject classification: 26D07, 26D10, 26D15, 26A33

1. Introduction

In conjunction with the development of differential and integral equations theory, integral inequalities which achieve explicit upper or lower bounds for unknown functions have gained great importance in mathematics. For this purpose a number of scientist have proposed numerous practical integral inequalities. On of the well-known integral inequalities was introduced by Čebysev [4] in 1882. Motivated by this inequality, scientist have found the answer many

*Corresponding Author

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question in the academia such as probability, statistical problems, numerical quadrature and transform theory. Then, in 1935, Grüss [6] introduced a practical and challenging inequality which provides an estimate of the difference between the integral of the product of two functions and the product of their integrals. In 1938, Ostrowski [15] was introduced an inequality associated with his name. Then various investigators have proposed different kind of Ostrowski type integral inequalities to achieve a variety of desired aims ([13],[14],[16],[18]). Meanwhile a number of mathematicians were involved in finding a way to take non-integer order of derivatives or integrals. As a result of these studies several methods have been introduced to solve fractional systems. Some of the most popular of these methods are Riemann-Liouville and Caputo definition ([5],[11],[12],[17]). Then Abdeljawad [1] and Khalil et. al. [10] defined the limit-based conformable derivative which is another type of fractional derivative and integrations. In more recent times a new local, limit-based definition of a conformable derivative has been introduced by Katugampola [9] in order to overcome some of difficulties which were given in [9].
The object of the present investigation is to obtain certain weighted Ostrowski, Čebysev and Grüss type integral inequalities involving the Katugampola conformable fractional integrals. The established results are a generalisation of some existing integral inequalities in the previous published studies.
The remainder of this work is organized as follows: In Section 2, the conformable derivatives are summarised, along with the Katugampola conformable fractional integrals type. Then weighted Ostrowski, Čebysev and Grüss type integral inequalities for conformable fractional integral are presented in Section 3, Section 4 and Section 5, respectively. Some conclusions and further directions of research are discussed in Section 6.

2. Preliminaries

In this section, we give some required properties of conformable fractional integrals introduced in detail in ([1]-[3],[7]-[10]). In this study, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon},\ D^{\alpha}(f)(0)=\lim_{t\to 0}D^{\alpha}(f)(t), \tag{2.1}$$

provided the limits exist (for detail see, [9]). If f is fully differentiable at t, then

$$D^{\alpha}(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{2.2}$$

A function f is $\alpha-$differentiable at a point $t\geq 0$ if the limit in (2.1) exists and is finite. This definition yields the following results;

Theorem 2.1. *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*

i. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, *for all* $a, b \in \mathbb{R}$,
ii. $D^{\alpha}(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
iii. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$,
iv. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$,
v. $D^{\alpha}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$
vi. $D^{\alpha}(f \circ g)(t) = f'(g(t)) D^{\alpha}(g)(t)$ *for* f *is differentiable at* $g(t)$.

Definition 2.2 (Conformable fractional integral)**.** Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_a^b f(x)\, d_{\alpha}x := \int_a^b f(x)\, x^{\alpha-1} dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$

Remark 2.3.

$$I_{\alpha}^{a}(f)(t) = I_{1}^{a}\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 2.4. *Let the conformable differential operator D^{α} be given as in (2.1), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f and g are α-differentiable as needed. Then*

i. $D^{\alpha}(\ln t) = t^{-\alpha}$ *for* $t > 0$
ii. $D^{\alpha}\left[\int_a^t f(t,s)\, d_{\alpha}s\right] = f(t,t) + \int_a^t D^{\alpha}[f(t,s)]\, d_{\alpha}s$
iii. $\int_a^b f(x)\, D^{\alpha}(g)(x)\, d_{\alpha}x = fg|_a^b - \int_a^b g(x)\, D^{\alpha}(f)(x)\, d_{\alpha}x$.

We can give the Hölder's inequality in conformable integral as follows:

Lemma 2.5. [19] *Let $f, g \in C[a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |f(x)g(x)|\, d_{\alpha}x \leq \left(\int_a^b |f(x)|^p\, d_{\alpha}x\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, d_{\alpha}x\right)^{\frac{1}{q}}.$$

Remark 2.6. If we take $p = q = 2$ in Lemma 2.5 , the we have the Cauchy-Schwartz inequality for conformable integral.

3. Weighted Ostrowski Type Inequality for Conformable Fractional Integral

In this section, we introduce the weighted vesion of Montgomery identity for comformable fractional integral. Then, we obtain the weighted Ostrowski inequality with the aid of that identity.

Firstly, we define the mapping $m(.,.):[a,b]^2 \to \mathbb{R}$ by

$$m(x,y)=\int_x^y w(s)d_\alpha s<\infty,\ x,y\in[a,b]$$

where $w:[a,b]\to\mathbb{R}$ is a non-negative function.

Now, we give a weighted Montgomery identity for conformable fractional integrals as follow:

Lemma 3.1. *Let $w:[a,b]\to\mathbb{R}$ be non-negative and $a_1,b_1,a,b\in\mathbb{R}$ with $0\le a\le a_1\le b_1\le b$, $a\neq b$, and let $f:[a,b]\to\mathbb{R}$ be $\alpha-$fractional differentiable for $\alpha\in(0,1]$. Then for $x\in[a,b]$ we have the following generalized weighted Montgomery identity for comformable fractional integrals*

$$m(a_1,b_1)f(x)+m(b_1,b)f(b)+m(a,a_1)f(a)-\int_a^b w(t)f(t)d_\alpha t=\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t \tag{3.1}$$

where

$$p(x,t)=\begin{cases} m(a_1,t), & t\in[a,x] \\ \\ m(b_1,t), & t\in(x,b]\,. \end{cases}$$

Proof. Using the integration by parts, we have

$$\begin{aligned}
&\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t \\
=\ & \int_a^x m(a_1,t)D_\alpha(f)(t)d_\alpha t+\int_x^b m(b_1,t)D_\alpha(f)(t)d_\alpha t \\
=\ & m(a_1,t)f(t)|_a^x-\int_a^x w(t)f(t)d_\alpha t+m(b_1,t)f(t)|_x^b-\int_x^b w(t)f(t)d_\alpha t \\
=\ & m(a_1,x)f(x)-m(a_1,a)f(a)+m(b_1,b)f(b)-m(b_1,x)f(x)-\int_a^b w(t)f(t)d_\alpha t \\
=\ & m(a_1,b_1)f(x)+m(a,a_1)f(a)+m(b_1,b)f(b)-\int_a^b w(t)f(t)d_\alpha t
\end{aligned}$$

which completes the proof. □

Corollary 3.2. *Under assumption of Lemma 3.1 with $a_1 = a$ and $b_1 = b$, we have the following weighted Montgomry identity for conformable fractional integrals*

$$m(a,b)f(x) - \int_a^b w(t)f(t)d_\alpha t = \int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t. \tag{3.2}$$

Remark 3.3. In Corollary 3.2, if we choose $w \equiv 1$, then we have the inequality

$$f(x) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t$$

which is given by Anderson in [2].

Theorem 3.4. *Suppose that the assumptions of Lemma 3.1 are satisfied, $\|D_\alpha(f)\|_\infty = \sup\limits_{x\in[a,b]} |D_\alpha(f)(x)| < \infty$, then we have the following weighted Ostrowski inequality*

$$\begin{aligned}&\left|m(a_1,b_1)f(x) + m(b_1,b)f(b) + m(a,a_1)f(a) - \int_a^b w(t)f(t)d_\alpha t\right| \qquad (3.3)\\ \leq\; & \frac{\|D_\alpha(f)\|_\infty \|w\|_\infty}{2\alpha^2}\left[(x^\alpha - a_1^\alpha)^2 + (a_1^\alpha - a^\alpha)^2 + (b_1^\alpha - x^\alpha)^2 + (b^\alpha - b_1^\alpha)^2\right].\end{aligned}$$

Proof. Taking modulus in Lemma 3.1, we have

$$\begin{aligned}&\left|m(a_1,b_1)f(x) + m(b_1,b)f(b) + m(a,a_1)f(a) - \int_a^b w(t)f(t)d_\alpha t\right|\\ \leq\; & \int_a^x |m(a_1,t)|\,|D_\alpha(f)(t)|\,d_\alpha t + \int_x^b |m(b_1,t)|\,|D_\alpha(f)(t)|\,d_\alpha t\\ \leq\; & \|D_\alpha(f)\|_{\infty,[a,x]}\,\|w\|_{\infty,[a,x]}\int_a^x\left|\int_{a_1}^t d_\alpha s\right| d_\alpha t\\ & + \|D_\alpha(f)\|_{\infty,[x,b]}\,\|w\|_{\infty,[x,b]}\int_x^b\left|\int_{b_1}^t d_\alpha s\right| d_\alpha t.\end{aligned}$$

Here, we get

$$\begin{aligned}\int_a^x \left|\int_{a_1}^t d_\alpha s\right| d_\alpha t &= \int_a^x \left|\frac{t^\alpha - a_1^\alpha}{\alpha}\right| d_\alpha t \\ &= \int_a^{a_1} \left(\frac{a_1^\alpha - t^\alpha}{\alpha}\right) d_\alpha t + \int_{a_1}^x \left(\frac{t^\alpha - a_1^\alpha}{\alpha}\right) d_\alpha t \\ &= \frac{1}{2\alpha^2}\left[(x^\alpha - a_1^\alpha)^2 + (a_1^\alpha - a^\alpha)^2\right]\end{aligned}$$

and similarly,

$$\begin{aligned}\int_x^b \left|\int_{b_1}^t d_\alpha s\right| d_\alpha t &= \int_x^b \left|\frac{b_1^\alpha - t^\alpha}{\alpha}\right| d_\alpha t \\ &= \int_x^{b_1} \left(\frac{b_1^\alpha - t^\alpha}{\alpha}\right) d_\alpha t + \int_{b_1}^b \left(\frac{t^\alpha - b_1^\alpha}{\alpha}\right) d_\alpha t \\ &= \frac{1}{2\alpha^2}\left[(b_1^\alpha - x^\alpha) + (b^\alpha - b_1^\alpha)^2\right].\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\left|m(a_1,b_1)f(x) + m(b_1,b)f(b) + m(a,a_1)f(a) - \int_a^b w(t)f(t)d_\alpha t\right| \\ \leq\ & \frac{\|D_\alpha(f)\|_{\infty,[a,x]}\,\|w\|_{\infty,[a,x]}}{2\alpha^2}\left[(x^\alpha - a_1^\alpha)^2 + (a_1^\alpha - a^\alpha)^2\right] \\ &+ \frac{\|D_\alpha(f)\|_{\infty,[x,b]}\,\|w\|_{\infty,[x,b]}}{2\alpha^2}\left[(b_1^\alpha - x^\alpha)^2 + (b^\alpha - b_1^\alpha)^2\right] \\ \leq\ & \frac{\|D_\alpha(f)\|_\infty\,\|w\|_\infty}{2\alpha^2}\left[(x^\alpha - a_1^\alpha)^2 + (b_1^\alpha - x^\alpha)^2 + (a_1^\alpha - a^\alpha)^2 + (b^\alpha - b_1^\alpha)^2\right]\end{aligned}$$

which completes the proof. □

Corollary 3.5. *Under assumption of Theorem 3.4 with $a_1 = a$ and $b_1 = b$, we have the following inequality*

$$\left|m(a,b)f(x) - \int_a^b w(t)f(t)d_\alpha t\right| \leq \frac{\|D_\alpha(f)\|_\infty\,\|w\|_\infty}{2\alpha^2}\left[(x^\alpha - a^\alpha)^2 + (b^\alpha - x^\alpha)^2\right].$$

Remark 3.6. In Corollary 3.5, if we choose $w \equiv 1$, then we have the inequality

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{\|D_\alpha(f)\|_\infty}{2\alpha\left(b^\alpha - a^\alpha\right)} \left[(x^\alpha - a^\alpha)^2 + (b^\alpha - x^\alpha)^2 \right]$$

which is given by Anderson in [2].

Theorem 3.7. *Suppose that the assumptions of Lemma 3.1 are satisfied, then we have the inequality*

$$\begin{aligned} & \left| m(a_1, b_1) f(x) + m(b_1, b) f(b) + m(a, a_1) f(a) - \int_a^b w(t) f(t) d_\alpha t \right| \\ \leq \quad & \frac{\|D_\alpha(f)\|_q \|w\|_p}{(p+1)^{\frac{1}{p}} \alpha^{\frac{p+1}{p}}} \left[(x^\alpha - a_1^\alpha)^{p+1} + (a^\alpha - a_1^\alpha)^{p+1} + (b_1^\alpha - x^\alpha)^{p+1} + (b_1^\alpha - b^\alpha)^{p+1} \right]^{\frac{1}{p}} \end{aligned}$$

where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|D_\alpha(f)\|_q$ *is defined by*

$$\|D_\alpha(f)\|_q = \left(\int_a^b |D_\alpha(f)(t)| \, d_\alpha t \right)^{\frac{1}{q}}.$$

Proof. Taking modulus in Lemma 3.1 and generalized Hölder's inequality (Lemma 2.5), we obtain

$$\begin{aligned} & \left| m(a_1, b_1) f(x) + m(b_1, b) f(b) + m(a, a_1) f(a) - \int_a^b w(t) f(t) d_\alpha t \right| \\ \leq \quad & \int_a^b |p(x,t)| \, |D_\alpha(f)(t)| \, d_\alpha t \\ \leq \quad & \left(\int_a^b |p(x,t)|^p \, d_\alpha t \right)^{\frac{1}{p}} \left(\int_a^b |D_\alpha(f)(t)|^q \, d_\alpha t \right)^{\frac{1}{q}} \\ = \quad & \|D_\alpha(f)\|_q \left[\int_a^x \left| \int_{a_1}^t w(s) d_\alpha s \right|^p d_\alpha t + \int_x^b \left| \int_{b_1}^t w(s) d_\alpha s \right|^p d_\alpha t \right]^{\frac{1}{p}} \\ \leq \quad & \|D_\alpha(f)\|_q \|w\|_p \left[\int_a^x \left| \frac{t^\alpha - a_1^\alpha}{\alpha} \right|^p d_\alpha t + \int_x^b \left| \frac{b_1^\alpha - t^\alpha}{\alpha} \right|^p d_\alpha t \right]^{\frac{1}{p}} \\ = \quad & \frac{\|D_\alpha(f)\|_q \|w\|_p}{(p+1)^{\frac{1}{p}} \alpha^{\frac{p+1}{p}}} \left[(x^\alpha - a_1^\alpha)^{p+1} + (a^\alpha - a_1^\alpha)^{p+1} + (b_1^\alpha - x^\alpha)^{p+1} + (b_1^\alpha - b^\alpha)^{p+1} \right]^{\frac{1}{p}} \end{aligned}$$

which completes the proof. □

Corollary 3.8. *Under assumption of Theorem 3.4 with $a_1 = a$ and $b_1 = b$, we have the following inequality*

$$\left| m(a,b)f(x) - \int_a^b w(t)f(t)d_\alpha t \right| \leq \frac{\|D_\alpha(f)\|_q \|w\|_p}{(p+1)^{\frac{1}{p}} \alpha^{\frac{p+1}{p}}} \left[(x^\alpha - a^\alpha)^{p+1} + (b^\alpha - x^\alpha)^{p+1} \right]^{\frac{1}{p}}.$$

Corollary 3.9. *Particularly, chosing $w \equiv 1$ in Corollary 3.8, we have the inequality*

$$\left| f(x) - \frac{1}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t \right| \leq \frac{\|D_\alpha(f)\|_q}{(p+1)^{\frac{1}{p}} (b^\alpha - a^\alpha)} \left[\frac{(x^\alpha - a^\alpha)^{p+1} + (b^\alpha - x^\alpha)^{p+1}}{\alpha} \right]^{\frac{1}{p}}.$$

4. Weighted Čebysev Type Inequality for Conformable Fractional Integral

Now, in Section 4, we present the weighted Čebysev inequality for conformable fractional integral with the help of Montgomery identity which was given in previous section.

Theorem 4.1. *Let $w : [a,b] \to \mathbb{R}$ be non-negative and let $a, b \in [0,\infty)$, and let $f, g : [a,b] \to \mathbb{R}$ be continuous functions with $\|D_\alpha(f)\|_\infty = \sup_{x\in[a,b]} |D_\alpha(f)(x)| < \infty$ and $\|D_\alpha(g)\|_\infty = \sup_{x\in[a,b]} |D_\alpha(g)(x)| < \infty$. Then, for $\alpha \in (0,1]$, we have the following weighted Čebysev inequality*

$$|T_\alpha(f,g;w)| \leq \frac{\|D_\alpha(f)\|_\infty \|D_\alpha(g)\|_\infty}{[m(a,b)]^3} \int_a^b w(x) (H(x))^2 d_\alpha x$$

where weighted Čebysev functional for conformable fractional integral $T_\alpha(f,g;w)$ is given by

$$\begin{aligned} T_\alpha(f,g;w) &= \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)d_\alpha t \qquad (4.1) \\ &\quad - \left(\frac{1}{m(a,b)} \int_a^b w(t)f(t)d_\alpha t \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(t)d_\alpha t \right) \end{aligned}$$

and

$$H(x) = \int_a^b |p(x,t)| \, d_\alpha t.$$

Proof. From Corollary 3.2, writing again the identity (3.2) for the functions $f(x)$ and $g(x)$, we have

$$f(x)-\frac{1}{m(a,b)}\int_a^b w(t)f(t)d_\alpha t=\frac{1}{m(a,b)}\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t \tag{4.2}$$

and

$$g(x)-\frac{1}{m(a,b)}\int_a^b w(t)g(t)d_\alpha t=\frac{1}{m(a,b)}\int_a^b p(x,t)D_\alpha(g)(t)d_\alpha t \tag{4.3}$$

where

$$p(x,t)=\begin{cases} m(a,t), & t\in[a,x] \\ m(b,t), & t\in(x,b]\,. \end{cases}$$

Multiplying the identities (4.2) and (4.3), we obtain

$$\begin{aligned} & f(x)g(x)-\frac{f(x)}{m(a,b)}\int_a^b w(t)g(t)d_\alpha t-\frac{g(x)}{m(a,b)}\int_a^b w(t)f(t)d_\alpha t \qquad (4.4)\\ & +\frac{1}{[m(a,b)]^2}\left(\int_a^b w(t)f(t)d_\alpha t\right)\left(\int_a^b w(t)g(t)d_\alpha t\right)\\ = & \frac{1}{[m(a,b)]^2}\left(\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t\right)\left(\int_a^b p(x,t)D_\alpha(g)(t)d_\alpha t\right). \end{aligned}$$

After multiplying both sides of (4.4) by $\frac{w(x)}{m(a,b)}$ integrating from a to b, then we obtain

$$\begin{aligned} & T_\alpha(f,g;w) \qquad (4.5)\\ = & \frac{1}{[m(a,b)]^3}\int_a^b w(x)\left(\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t\right)\left(\int_a^b p(x,t)D_\alpha(g)(t)d_\alpha t\right)d_\alpha x. \end{aligned}$$

Taking the modulus in (4.5), we get

$$
\begin{aligned}
& |T_\alpha(f,g;w)| \\
\leq\ & \frac{1}{[m(a,b)]^3}\int\limits_a^b w(x)\left(\int\limits_a^b |p(x,t)|\,|D_\alpha(f)(t)|\,d_\alpha t\right)\left(\int\limits_a^b |p(x,t)|\,|D_\alpha(g)(t)|\,d_\alpha t\right)d_\alpha x \\
\leq\ & \frac{\|D_\alpha(f)\|_\infty\,\|D_\alpha(g)\|_\infty}{[m(a,b)]^3}\int\limits_a^b w(x)\left(\int\limits_a^b |p(x,t)|\,d_\alpha t\right)^2 d_\alpha x \\
=\ & \frac{\|D_\alpha(f)\|_\infty\,\|D_\alpha(g)\|_\infty}{[m(a,b)]^3}\int\limits_a^b w(x)\,(H(x))^2\,d_\alpha x.
\end{aligned}
$$

□

Corollary 4.2. *Under assumption of Theorem 4.1 with $w \equiv 1$, we have the following Čebysev inequality*

$$
|T_\alpha(f,g)| \leq \frac{7}{60}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^2 \|D_\alpha(f)\|_\infty\,\|D_\alpha(g)\|_\infty
$$

where

$$
\begin{aligned}
T_\alpha(f,g) = & \frac{\alpha}{b^\alpha - a^\alpha}\int\limits_a^b w(t)f(t)g(t)d_\alpha t \\
& -\left(\frac{\alpha}{b^\alpha - a^\alpha}\int\limits_a^b w(t)f(t)d_\alpha t\right)\left(\frac{\alpha}{b^\alpha - a^\alpha}\int\limits_a^b w(t)g(t)d_\alpha t\right).
\end{aligned}
\tag{4.6}
$$

Proof. For $w \equiv 1$, we have

$$
m(a,b) = \frac{b^\alpha - a^\alpha}{\alpha}
$$

and

$$
\begin{aligned}
H(x) &= \int\limits_a^b \left(\frac{t^\alpha - a^\alpha}{\alpha}\right)d_\alpha t + \int\limits_a^b \left(\frac{b^\alpha - t^\alpha}{\alpha}\right)d_\alpha t \\
&= \frac{(x^\alpha - a^\alpha)^2 + (b^\alpha - x^\alpha)^2}{2\alpha^2} \\
&= \frac{1}{\alpha^2}\left[\left(x^\alpha - \frac{a^\alpha + b^\alpha}{2}\right)^2 + \frac{(b^\alpha - a^\alpha)^2}{4}\right].
\end{aligned}
$$

Thn we have

$$\int_a^b (H(x))^2 \, d_\alpha x = \frac{7}{60}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^5$$

which completes the proof. □

Theorem 4.3. *Suppose that the assumptions of Theorem 4.1 are satisfied, then we have the following weighted Čebysev inequality for conformable fractional integral*

$$\begin{aligned} |T_\alpha(f,g;w)| &\leq \frac{1}{2\left[m(a,b)\right]^2}\left[\|D_\alpha(f)\|_\infty \int_a^b w(x)\,|g(x)|\,H(x)d_\alpha x \right. \\ &\quad \left. + \|D_\alpha(g)\|_\infty \int_a^b w(x)\,|f(x)|\,H(x)d_\alpha x\right]. \end{aligned}$$

Proof. Multiplying the identities (4.2) and (4.3) by $w(x)g(x)$ and $w(x)f(x)$ respectively, then combining resulting identity, we establish

$$\begin{aligned} &2w(x)f(x)g(x) - \frac{w(x)g(x)}{m(a,b)}\int_a^b w(t)f(t)d_\alpha t - \frac{w(x)f(x)}{m(a,b)}\int_a^b w(t)g(t)d_\alpha t \qquad (4.7) \\ &= \frac{w(x)g(x)}{m(a,b)}\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t + \frac{w(x)f(x)}{m(a,b)}\int_a^b p(x,t)D_\alpha(g)(t)d_\alpha t. \end{aligned}$$

Integrating the inequality (4.7), we have

$$\begin{aligned} &T_\alpha(f,g;w) \\ &= \frac{1}{2\left[m(a,b)\right]^2}\left[\int_a^b w(x)g(x)\left(\int_a^b p(x,t)D_\alpha(f)(t)d_\alpha t\right)d_\alpha x \right. \\ &\quad \left. + \int_a^b w(x)f(x)\left(\int_a^b p(x,t)D_\alpha(g)(t)d_\alpha t\right)d_\alpha x\right]. \end{aligned}$$

Then, it follows that

$$
\begin{aligned}
& |T_\alpha(f,g;w)| \\
\leq\ & \frac{1}{2\left[m(a,b)\right]^2}\left[\int_a^b w(x)\left|g(x)\right|\left(\int_a^b \left|p(x,t)\right|\left|D_\alpha(f)(t)\right| d_\alpha t\right) d_\alpha x\right. \\
& \left. + \int_a^b w(x)\left|f(x)\right|\left(\int_a^b \left|p(x,t)\right|\left|D_\alpha(g)(t)\right| d_\alpha t\right) d_\alpha x\right] \\
\leq\ & \frac{1}{2\left[m(a,b)\right]^2}\left[\left\|D_\alpha(f)\right\|_\infty \int_a^b w(x)\left|g(x)\right| H(x) d_\alpha x\right. \\
& \left. + \left\|D_\alpha(g)\right\|_\infty \int_a^b w(x)\left|f(x)\right| H(x) d_\alpha x\right].
\end{aligned}
$$

This completes the proof. $\square$

Corollary 4.4. *Under assumption of Theorem 4.3 with $w \equiv 1$, we have the following Čebysev inequality*

$$
|T_\alpha(f,g)| \leq \frac{1}{2}\left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2\left[\left\|D_\alpha(f)\right\|_\infty \int_a^b \left|g(x)\right| H(x) d_\alpha x + \left\|D_\alpha(g)\right\|_\infty \int_a^b \left|f(x)\right| H(x) d_\alpha x\right].
$$

5. Weighted Grüss Type Inequality for Conformable Fractional Integral

Finally, we provide the Grüss type inequality for conformable fractional integral following similar steps in previous sections.

Theorem 5.1. *Let $w : [a,b] \to \mathbb{R}$ be non-negative function with $a, b \in [0,\infty)$, and let $f, g : [a,b] \to \mathbb{R}$ be $\alpha-$fractional integrable functions with $\alpha \in (0,1]$ and*

$$
m_1 \leq f(x) \leq M_1,\ m_2 \leq g(x) \leq M_2 \text{ for all } x \in [a,b].
$$

Then we have the following weighted Grüss inequality

$$
|T_\alpha(f,g;w)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{5.1}
$$

where $T_\alpha(f,g;w)$ is given by (4.1).

Proof. We have

$$\begin{aligned}
&\int_a^b \int_a^b [f(x) - f(y)] [g(x) - g(y)] w(x)w(y) d_\alpha x d_\alpha y \qquad (5.2)\\
= &\int_a^b \int_a^b [f(x)g(x) - f(x)g(y) - f(y)g(x) + f(y)g(y)] w(x)w(y) d_\alpha x d_\alpha y\\
= &\ 2m(a,b) \int_a^b w(x)f(x)g(x) d_\alpha x - 2 \left(\int_a^b w(x)f(x) d_\alpha x \right) \left(\int_a^b w(x)g(x) d_\alpha x \right)\\
= &\ 2 [m(a,b)]^2 T_\alpha(f, g; w).
\end{aligned}$$

That is,

$$\begin{aligned}
&T_\alpha(f, g; w) \qquad (5.3)\\
= &\frac{1}{2 [m(a,b)]^2} \int_a^b \int_a^b [f(x) - f(y)] [g(x) - g(y)] w(x)w(y) d_\alpha x d_\alpha y.
\end{aligned}$$

Appling Cauchy-Schwartz inequality (Remark 2.6), we obtain

$$\begin{aligned}
&\left[\frac{1}{2 [m(a,b)]^2} \int_a^b \int_a^b [f(x) - f(y)] [g(x) - g(y)] w(x)w(y) d_\alpha x d_\alpha y \right]^2 \qquad (5.4)\\
\leq &\left(\frac{1}{2 [m(a,b)]^2} \int_a^b \int_a^b [f(x) - f(y)]^2 w(x)w(y) d_\alpha x d_\alpha y \right)\\
&\times \left(\frac{1}{2 [m(a,b)]^2} \int_a^b \int_a^b [g(x) - g(y)]^2 w(x)w(y) d_\alpha x d_\alpha y \right)\\
= &\left[\frac{1}{m(a,b)} \int_a^b w(x)f^2(x) d_\alpha x - \left(\frac{1}{m(a,b)} \int_a^b w(x)f(x) d_\alpha x \right)^2 \right]\\
&\times \left[\frac{1}{m(a,b)} \int_a^b w(x)g^2(x) d_\alpha x - \left(\frac{1}{m(a,b)} \int_a^b w(x)g(x) d_\alpha x \right)^2 \right].
\end{aligned}$$

It is easy to observe that

$$\begin{aligned}
&\frac{1}{m(a,b)}\int_a^b w(x)f^2(x)d_\alpha x-\left(\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x\right)^2\\
=&\left(M_1-\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x-m_1\right)\\
&-\frac{1}{m(a,b)}\int_a^b [M_1-f(x)]\,[f(x)-m_1]\,w(x)d_\alpha x.
\end{aligned}$$

Since $[M_1-f(x)]\,[f(x)-m_1]\geq 0$ for each $x\in[a,b]$, then we get

$$\begin{aligned}
&\frac{1}{m(a,b)}\int_a^b w(x)f^2(x)d_\alpha x-\left(\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x\right)^2 \qquad (5.5)\\
\leq&\left(M_1-\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x-m_1\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{m(a,b)}\int_a^b w(x)g^2(x)d_\alpha x-\left(\frac{1}{m(a,b)}\int_a^b w(x)g(x)d_\alpha x\right)^2 \qquad (5.6)\\
\leq&\left(M_2-\frac{1}{m(a,b)}\int_a^b w(x)g(x)d_\alpha x\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)g(x)d_\alpha x-m_2\right).
\end{aligned}$$

Using (5.5) and (5.6) in (5.4), we get the following inequality

$$\begin{aligned}
&\left[\frac{1}{2\,[m(a,b)]^2}\int_a^b\int_a^b [f(x)-f(y)]\,[g(x)-g(y)]\,w(x)w(y)d_\alpha x d_\alpha y\right]^2\\
\leq&\left(M_1-\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)f(x)d_\alpha x-m_1\right)\\
&\times\left(M_2-\frac{1}{m(a,b)}\int_a^b w(x)g(x)d_\alpha x\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)g(x)d_\alpha x-m_2\right).
\end{aligned}$$

Now, using the elementary inequality for real numbers

$$pq\leq\frac{1}{4}\,(p+q)^2\,,\quad p,q\in\mathbb{R}$$

we get

$$\left[\frac{1}{2\left[m(a,b)\right]^2}\int_a^b\int_a^b [f(x)-f(y)]\,[g(x)-g(y)]\,w(x)w(y)d_\alpha x d_\alpha y\right]^2$$

$$\leq \frac{1}{16}(M_1-m_1)^2(M_2-m_2)^2$$

which completes the proof. To prove the sharpness of (5.1), let choose

$$w(x)=1 \text{ and } f(x)=g(x)=\begin{cases} -1, & a\leq x<\left(\frac{a^\alpha+b^\alpha}{2}\right)^{\frac{1}{\alpha}} \\ \\ 1, & \left(\frac{a^\alpha+b^\alpha}{2}\right)^{\frac{1}{\alpha}}\leq x\leq b, \end{cases}$$

then we have

$$\frac{1}{m(a,b)}\int_a^b w(x)g(x)f(x)d_\alpha x=\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b g(x)f(x)d_\alpha x=1,$$

$$\int_a^b w(x)f(x)d_\alpha x=\int_a^b w(x)g(x)d_\alpha x=0$$

and

$$M_1-m_1=M_2-m_2=2$$

which the equality (5.1) is realized. □

Remark 5.2. Under assumption of Theorem 5.1 with $w\equiv 1$, we have the following Grüss inequality

$$|T_\alpha(f,g)|\leq\frac{1}{4}(M_1-m_1)(M_2-m_2)$$

which is given by Anderson in [2].

Theorem 5.3. *Let $w:[a,b]\to\mathbb{R}$ be non-negative function with $a,b\in[0,\infty)$, and let $f,g:[a,b]\to\mathbb{R}$ be $\alpha-$fractional differantiable functions on (a,b) with $\alpha\in(0,1]$. Then we have the following inequalties*

$$|T_\alpha(f,g;w)| \tag{5.7}$$

$$\leq \frac{1}{2\left[m(a,b)\right]^2}\left(\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right|\left|\int_x^y |D_\alpha(f)(t)|^p\,d_\alpha t\right|d_\alpha x d_\alpha y\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right|\left|\int_x^y |D_\alpha(g)(s)|^q\,d_\alpha s\right|d_\alpha x d_\alpha y\right)^{\frac{1}{q}}$$

$$\leq \frac{\|D_\alpha(f)\|_p\,\|D_\alpha(g)\|_q}{2\left[m(a,b)\right]^2}\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right|d_\alpha x d_\alpha y$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_x^y\int_x^y \left|D_\alpha(f)(t)D_\alpha(g)(s)\right| d_\alpha t d_\alpha s\right| \qquad (5.8)\\
\leq\; & \left|\int_x^y\int_x^y \left|D_\alpha(f)(t)\right|^p d_\alpha t d_\alpha s\right|^{\frac{1}{p}} \left|\int_x^y\int_x^y \left|D_\alpha(g)(s)\right|^q d_\alpha t d_\alpha s\right|^{\frac{1}{q}}\\
=\; & \left|\frac{y^\alpha-x^\alpha}{\alpha}\right|^{\frac{1}{p}} \left|\int_x^y \left|D_\alpha(f)(t)\right|^p d_\alpha t\right|^{\frac{1}{p}} \left|\frac{y^\alpha-x^\alpha}{\alpha}\right|^{\frac{1}{q}} \left|\int_x^y \left|D_\alpha(g)(s)\right|^q d_\alpha s\right|^{\frac{1}{q}}\\
=\; & \left|\frac{y^\alpha-x^\alpha}{\alpha}\right| \left|\int_x^y \left|D_\alpha(f)(t)\right|^p d_\alpha t\right|^{\frac{1}{p}} \left|\int_x^y \left|D_\alpha(g)(s)\right|^q d_\alpha s\right|^{\frac{1}{q}}.
\end{aligned}$$

On the other hand, from (5.3) and (5.8), we have

$$\begin{aligned}
&\left|T_\alpha(f,g;w)\right|\\
\leq\; & \frac{1}{2\left[m(a,b)\right]^2}\int_a^b\int_a^b \left|f(x)-f(y)\right|\left|g(x)-g(y)\right| w(x)w(y)d_\alpha x d_\alpha y\\
=\; & \frac{1}{2\left[m(a,b)\right]^2}\int_a^b\int_a^b w(x)w(y)\left|\int_x^y\int_x^y \left|D_\alpha(f)(t)D_\alpha(g)(s)\right| d_\alpha t d_\alpha s\right| d_\alpha x d_\alpha y\\
\leq\; & \frac{1}{2\left[m(a,b)\right]^2}\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right| \left|\int_x^y \left|D_\alpha(f)(t)\right|^p d_\alpha t\right|^{\frac{1}{p}} \left|\int_x^y \left|D_\alpha(g)(s)\right|^q d_\alpha s\right|^{\frac{1}{q}} d_\alpha x d_\alpha y.
\end{aligned}$$

Applying again Hölder inequality, we obtain

$$\begin{aligned}
&\left|T_\alpha(f,g;w)\right|\\
\leq\; & \frac{1}{2\left[m(a,b)\right]^2}\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right| \left|\int_x^y \left|D_\alpha(f)(t)\right|^p d_\alpha t\right|^{\frac{1}{p}} \left|\int_x^y \left|D_\alpha(g)(s)\right|^q d_\alpha s\right|^{\frac{1}{q}} d_\alpha x d_\alpha y\\
\leq\; & \frac{1}{2\left[m(a,b)\right]^2}\left(\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right| \left|\int_x^y \left|D_\alpha(f)(t)\right|^p d_\alpha t\right| d_\alpha x d_\alpha y\right)^{\frac{1}{p}}\\
&\times\left(\int_a^b\int_a^b w(x)w(y)\left|\frac{y^\alpha-x^\alpha}{\alpha}\right| \left|\int_x^y \left|D_\alpha(g)(s)\right|^q d_\alpha s\right| d_\alpha x d_\alpha y\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof of the first inequality in (5.7).

Using the facts that

$$\left| \int_x^y |D_\alpha(f)(t)|^p \, d_\alpha t \right| \leq \|D_\alpha(f)\|_p^p \text{ and } \left| \int_x^y |D_\alpha(g)(s)|^q \, d_\alpha s \right| \leq \|D_\alpha(g)\|_q^q$$

we can easily obtain the the proof of the second inequality in (5.7). □

Corollary 5.4. *Under assumption of Theorem 5.3 with $w \equiv 1$, we have the following inequalities*

$$\begin{aligned}
& |T(f,g)| \\
\leq\ & \frac{1}{2}\left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2 \left(\int_a^b \int_a^b \left| \frac{y^\alpha - x^\alpha}{\alpha} \right| \left| \int_x^y |D_\alpha(f)(t)|^p \, d_\alpha t \right| d_\alpha x d_\alpha y \right)^{\frac{1}{p}} \\
& \times \left(\int_a^b \int_a^b \left| \frac{y^\alpha - x^\alpha}{\alpha} \right| \left| \int_x^y |D_\alpha(g)(s)|^q \, d_\alpha s \right| d_\alpha x d_\alpha y \right)^{\frac{1}{q}} \\
\leq\ & \frac{b^\alpha - a^\alpha}{6\alpha} \left\| D_\alpha(f) \right\|_p \left\| D_\alpha(g) \right\|_q .
\end{aligned}$$

6. Concluding Remark

The purpose of the current study was to make a generalization of some integral inequalities with the help of Katugampola conformable fractional integrals. The results of this research show parallelism with the previous studies. A further study could be assess by using the two independent variables.

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