Iranian Journal of Mathematical Sciences and Informatics
Vol. 16, No. 1 (2021), pp 55-64
DOI: 10.29252/ijmsi.16.1.55

Wijsman Statistical Convergence of Double Sequences of Set

Fatih Nuray, Erdinç Dündar*, Uğur Ulusu

Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, Afyonkarahisar, Turkey.

E-mail: fnuray@aku.edu.tr
E-mail: edundar@aku.edu.tr
E-mail: ulusu@aku.edu.tr

Abstract. In this paper, we study the concepts of Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationship between them.

Keywords: Statistical convergence, Double sequence of sets, Wijsman convergence, Hausdorff convergence.

2010 Mathematics subject classification: 40A05, 40A35.

1. Introduction

Hill [12] was the first who applied methods of functional analysis to double sequences. Also, Kull [14] applied methods of functional analysis of matrix maps of double sequences. A lot of useful developments of double sequences in summability methods in [1, 8, 13, 15, 21, 30].

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [22]. This concept was extended to the double sequences by Mursaleen and Edely [16]. çakan and Altay [7] presented multidimensional analogues of the results presented by Fridy and Orhan [11].

*Corresponding Author

Received 05 October 2017; Accepted 27 July 2018

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [3–5, 17, 27–29]). Nuray and Rhoades [17] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [26] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Ulusu and Nuray [27] introduced the concept of Wijsman strongly lacunary summability for set sequences and discused its relation with Wijsman strongly Cesàro summability. Nuray et. al. [18] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigated the relationship between them. Talo and Sever [23] examined the relationship between Kuratowski statistical convergence and Hausdorff statistical convergence.

In this paper, we study the concepts of Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationship between them.

2. DEFINITIONS AND NOTATIONS

Now, we recall the basic definitions and concepts (See [1–5, 17–19, 21, 23, 24, 27–29]).

For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper, we let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X.

The sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case, we write $W - \lim A_k = A$.

The sequence $\{A_k\}$ is Hausdorff convergent to A if

$$\lim_{k \to \infty} \sup_{x \in X} |d(x, A_k) - d(x, A)| = 0$$

In this case, we write $H - \lim A_k = A$.

The sequence $\{A_k\}$ is Wijsman statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

The sequence $\{A_k\}$ is Wijsman statistically Cauchy if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x, A_k) - d(x, A_N)| \ge \varepsilon\}| = 0.$$

The sequence $\{A_k\}$ is Hausdorff statistically convergent to A if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \sup_{x\in X} |d(x, A_k) - d(x, A)| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim_H A_k = A$ or $A_k \to A(HS)$.

The sequence $\{A_k\}$ is Hausdorff statistically Cauchy if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that for each $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \sup_{x\in X} |d(x, A_k) - d(x, A_N)| \ge \varepsilon\}| = 0.$$

A double sequence $x = (x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case, we write

$$P - \lim_{k,j\to\infty} x_{kj} = L \quad or \quad \lim_{k,j\to\infty} x_{kj} = L.$$

A double sequence $x = (x_{kj})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{kj}| < M$ for all $k, j \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{k,j} |x_{kj}| < \infty.$$

Throughout the paper, we suppose A, A_{kj} be any non-empty closed subsets of X.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \quad or \quad \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$. In this case, we write $W_2 - \lim A_{kj} = A$.

The double sequence $\{A_{kj}\}$ is said to be Wijsman Cesàro summable to A if $\{d(x, A_{kj})\}$ Cesaro summable to $\{d(x, A)\}$; that is, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj}) = d(x, A).$$

In this case, we write $A_{kj} \stackrel{(W_2\sigma_1)}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly Cesàro summable to A if $\{d(x, A_{kj})\}$ strongly Cesàro summable to $\{d(x, A)\}$; that is, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2\sigma_1]}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly p-Cesàro summable to A if $\{d(x, A_{kj})\}$ strongly p-Cesàro summable to $\{d(x, A)\}$; that is, for each p positive real number and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)|^p = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2\sigma_p]}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is Hausdorff statistically convergent to A if for each $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{k \leq m, j \leq n : \sup_{x\in X} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st_2 - \lim_H A_{kj} = A$.

3. Main results

Definition 3.1. The double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0,$$

that is, $|d(x, A_{kj}) - d(x, A)| < \varepsilon$, a.a. (k,j). In this case, we write $st_2 - \lim_W A_k = A$.

Example 3.2. Let $X = \mathbb{R}^2$ and double sequence $\{A_{kj}\}$ be following sequence:

$$A_{kj} = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y)^2 = \frac{1}{kj}\right\}, & \text{if k and j is a square integer} \\ \{(0, 0)\}, & \text{otherwise.} \end{cases}$$

This double sequence is bounded and Wijsman statistically convergent to the set $A = \{(0, 0)\}$ but it is not Wijsman convergent.

Example 3.3. Let $X = \mathbb{R}^2$ and double sequence $\{A_{kj}\}$ be following sequence:

$$A_{kj} = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : (x-k)^2 + (y-j)^2 = 1\right\}, & \text{if k and j is a square integer} \\ \{(0, 0)\}, & \text{otherwise.} \end{cases}$$

This double sequence is Wijsman statistically convergent to the set $A = \{(0, 0)\}$ but it is not Wijsman convergent.

If a double sequence of sets $\{A_{kj}\}$ is Wijsman statistically convergent to the set A, then $\{A_{kj}\}$ need not be Wijsman convergent. Also, it is not necessary be bounded.

Example 3.4. Let $X = \mathbb{R}^2$ and double sequence $\{A_{kj}\}$ be following sequence:

$$A_{kj} = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = kj\right\}, & \text{if k and j is a square integer} \\ \{(2, 2)\}, & \text{otherwise.} \end{cases}$$

This double sequence is Wijsman statistically convergent to the set $A = \{(2,2)\}$ but it is neither Wijsman convergent nor bounded.

Theorem 3.5. A double set sequence $\{A_{kj}\}$ is Wijsman statistically convergent to the set A if and only if there exists a subset $K = \{(k,j)\} \subset \mathbb{N} \times \mathbb{N}$, $k, j = 1, 2, ...$, such that

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{k \le m, j \le n : (k,j) \in K\}| = 1 \text{ and } \lim_{\substack{k,j\to\infty \\ (k,j)\in K}} d(x, A_{kj}) = d(x, A),$$

for each $x \in X$.

Proof. Let $\{A_{kj}\}$ be Wijsman statistical convergent to A. For each $x \in X$, put

$$K(r,x) = \left\{k \le m, j \le n : |d(x, A_{kj}) - d(x, A)| \ge \frac{1}{r}\right\}$$

and

$$M(r,x) = \left\{k \le m, j \le n : |d(x, A_{kj}) - d(x, A)| < \frac{1}{r}\right\} \qquad (r = 1, 2, ...).$$

Then,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{k \le m, j \le n : |d(x, A_{kj}) - d(x, A) \ge \frac{1}{r}\right\}\right| = 0$$

and

$$M(1,x) \supset M(2,x) \supset \cdots \supset M(i,x) \supset M(i+1,x) \cdots \tag{3.1}$$

and

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{k \le m, j \le n : |d(x, A_{kj}) - d(x, A) < \frac{1}{r}\right\}\right| = 1 \qquad (r = 1, 2, ...). \tag{3.2}$$

Now, we have to show that for $(k,j) \in M(r,x)$, $\{A_{kj}\}$ is Wijsman convergent to A. Suppose that $\{A_{kj}\}$ is not Wijsman convergent to A. Therefore, there is a number $\varepsilon > 0$ such that for each $x \in X$,

$$|d(x, A_{kj}) - d(x, A)| \ge \varepsilon$$

for infinitely many terms. For each $x \in X$, let

$$M(\varepsilon, x) = \{k \le m, j \le n : |d(x, A_{kj}) - d(x, A) < \varepsilon\} \text{ and } \varepsilon > \frac{1}{r} \quad (r = 1, 2, ...).$$

Then,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{k \le m, j \le n : |d(x, A_{kj}) - d(x, A)| < \varepsilon| = 0,$$

and by (3.1), $M(r,x) \subset M(\varepsilon, x)$. Hence

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{k \le m, j \le n : |d(x, A_{kj}) - d(x, A) < \frac{1}{r}\right\}\right| = 0$$

which contradicts (3.2). Therefore, $\{A_{kj}\}$ is Wijsman convergent to A.

Conversely, suppose that there exists a subset $K = \{(k,j)\} \subset \mathbb{N} \times \mathbb{N}$ such that

$$\lim_{k,j\to\infty} \frac{1}{kj}|K| = 1$$

and

$$\lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$$

i.e., there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ and for each $x \in X$,

$$|d(x, A_{kj} - d(x, A)| < \varepsilon, \qquad \forall k, j \geq N.$$

Now,

$$K(\varepsilon, x) = \{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}$$

$$\subseteq \mathbb{N} \times \mathbb{N} - \{(k_{N+1}, j_{N+1}), (k_{N+2}, j_{N+2}), ...\}.$$

Therefore,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon| \leq 1 - 1 = 0.$$

Hence, $\{A_{kj}\}$ is Wijsman statistically convergent to A. □

If $\{A_{kj}\}$ is convergent but unbounded then $\{A_{kj}\}$ is Wijsman statistically convergent but $\{A_{kj}\}$ need not be Wijsman Cesàro summable nor strongly Wijsman Cesàro summable.

Example 3.6. Let $X = \mathbb{R}^2$ and double sequence $\{A_{kj}\}$ be defined as:

$$A_{kj} = \begin{cases} \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = k\} & , \ j = 1, \text{ for all k}, \\ \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = j\} & , \ k = 1, \text{ for all j}, \\ \{(0,0)\} & , \ otherwise. \end{cases}$$

Then, $\{A_{kj}\}$ is Wijsman convergent to $\{(0,0)\}$ but

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj})$$

does not tend to a finite limit. Hence $\{A_{kj}\}$ is not Wijsman Cesàro summable. Also, $\{A_{kj}\}$ is not strongly Cesàro summable but

$$\lim_{m,n\to\infty} \frac{1}{mn}|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, \{(0,0)\})| \geq \varepsilon\}|$$
$$= \lim_{m,n\to\infty} \frac{m+n-1}{mn} = 0,$$

that is, $\{A_{kj}\}$ is Wijsman statistically convergent to $\{(0,0)\}$.

Theorem 3.7. If $\{A_{kj}\}$ is Wijsman strongly p-Cesàro summable to A, then it is Wijsman statistically convergent to A.

Proof. For any $\{A_{kj}\}$, fix an $\varepsilon > 0$. Then

$$\sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)|^p \geq \varepsilon^p |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}|$$

and it follows that if $\{A_{kj}\}$ is Wijsman strongly $p-$Cesàro summable to A then $\{A_{kj}\}$ is Wijsman statistically convergent to A. □

Theorem 3.8. If $\{A_{kj}\}$ is Wijsman statistically convergent to A and bounded, then it is Wijsman strongly p-Cesàro summable to A.

Proof. Let $\{A_{kj}\}$ be bounded and Wijsman statistically convergent to A. Since $\{A_{kj}\}$ is bounded, set

$$\sup_{k,j} \{d(x, A_{kj})\} + d(x, A) = M.$$

Since $\{A_{kj}\}$ is Wijsman statistically convergent to A, for given $\varepsilon > 0$ we can select N_ε such that for each $x \in X$,

$$\frac{1}{mn} \left| \left\{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}} \right\} \right| < \frac{\varepsilon}{2M^p}$$

for all $m, n > N_\varepsilon$ and set

$$L(\varepsilon, x) = \left\{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}} \right\}.$$

Then,

$$\begin{aligned} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)|^p &= \frac{1}{mn} \left(\sum_{\substack{k \leq m, j \leq n \\ k,j \in L(\varepsilon, x)}} |d(x, A_{kj}) - d(x, A)|^p \right. \\ &\quad \left. + \sum_{\substack{k \leq m, j \leq n \\ k,j \notin L(\varepsilon, x)}} |d(x, A_{kj}) - d(x, A)|^p \right) \\ &< \frac{1}{mn} . \frac{mn.\varepsilon}{2M^p} M^p + \frac{1}{mn} . \frac{mn.\varepsilon}{2} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $\{A_{kj}\}$ is Wijsman strongly p-summable to A. □

Definition 3.9. The double sequence $\{A_{kj}\}$ is Wijsman statistically Cauchy if for each $\varepsilon > 0$ there exist numbers $r = r(\varepsilon), s = s(\varepsilon) \in \mathbb{N}$ such that for each $x \in X$,

$$\lim_{m,n \to \infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A_{rs})| \geq \varepsilon\}| = 0.$$

Theorem 3.10. A double sequence $\{A_{kj}\}$ is Wijsman statistically convergent if and only if $\{A_{kj}\}$ is Wijsman statistically Cauchy.

Proof. Suppose that $st_2-\lim_W A_{kj}=A$ and $\varepsilon>0$. Then, we get

$$|d(x,A_{kj})-d(x,A)|\leq\frac{\varepsilon}{2},\ \text{a.a. } (k,j).$$

If r,s is chosen so that $|d(x,A_{rs})-d(x,A)|\leq\frac{\varepsilon}{2}$, then we have for each $x\in X$,

$$\begin{aligned}|d(x,A_{kj})-d(x,A_{rs})| &\leq |d(x,A_{kj})-d(x,A)|+|d(x,A_{rs})-d(x,A)|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\quad \text{a.a. (k,j)}.\end{aligned}$$

Hence, $\{A_{kj}\}$ is a Wijsman statistically Cauchy double sequence.

Conversely let A_{kj} is a Wijsman statistically Cauchy double sequence, then choose r,s so that the band $I=[d(x,A_{rs})-1,d(x,A_{rs})+1]$ contains $d(x,A_{kj})$ a.a. (k,j) for each (fixed) $x\in X$. Now choose r_1,s_1 such that $I'=[d(x,A_{r_1s_1})-\frac{1}{2},d(x,A_{r_1s_1})+\frac{1}{2}]$ contains $d(x,A_{kj})$ a.a. (k,j) for each (fixed) $x\in X$. We assert that $I_1=I\cap I'$ contains $d(x,A_{kj})$ a.a. (k,j) for each (fixed) $x\in X$, then

$$\begin{aligned}\{k\leq m,j\leq n: d(x,A_{kj})\notin I_1\} &= \{k\leq m,j\leq n: d(x,A_{kj})\notin I\}\\ &\quad \cup\{k\leq m,j\leq n: d(x,A_{kj})\notin I'\}\end{aligned}$$

for each (fixed) $x\in X$, so

$$\begin{aligned}&\lim_{m,n\to\infty}\frac{1}{mn}\{k\leq m,j\leq n: d(x,A_{kj})\notin I_1\}\\ &\qquad=\lim_{m,n\to\infty}\frac{1}{mn}\{k\leq m,j\leq n: d(x,A_{kj})\notin I\}\\ &\qquad\quad+\lim_{m,n\to\infty}\frac{1}{mn}\{k\leq m,j\leq n: d(x,A_{kj})\notin I'\}\end{aligned}$$

for each (fixed) $x\in X$.

Therefore I_1 is a closed band of height less than or equal to 1 that contains $d(x,A_{kj})$ a.a. (k,j) for every $x\in X$. Now we proceed by choosing r_2,s_2 so that $I''=[d(x,A_{r_2s_2})-\frac{1}{4},d(x,A_{r_2s_2})+\frac{1}{4}]$ contains $d(x,A_{kj})$ a.a. (k,j) and by the preceding argument $I_2=I_1\cap I''$ contains $d(x,A_{kj})$ a.a. (k,j) for each (fixed) $x\in X$ and I_2 has height less than or equal to $\frac{1}{2}$. Continuing inductively we construct a sequence $\{I_p\}_{p=1}^{\infty}$ of closed band such that for each p, $I_p\supseteq I_{p+1}$, the height of I_p is not greater than 2^{1-p} and $d(x,A_{kj})\in I_p$ a.a. (k,j) for each (fixed) $x\in X$. Thus, there exists $d(x,A)$, defined on X, that $\{d(x,A)\}$ is equal to $\bigcap_{p=1}^{\infty}I_p$.

Now, we show that $\{d(x,A_{kj})\}$ is statistically convergent to $d(x,A)$ on X. Let $\varepsilon>0$ be given, then there exist q such that $\varepsilon>2^{1-q}$. Then from the above construction it follows that $d(x,A_{kj})\in I_q$ a.a. (k,j) for each (fixed) $x\in X$.

We have

$$\frac{1}{mn}\left|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}\right|$$
$$\leq \frac{1}{mn}\left|\{k \leq m, j \leq n : d(x, A_{kj}) \notin I_q\}\right| \to 0.$$

for each (fixed) $x \in X$. Thus, the proof of the theorem is completed.

□

Theorem 3.11. If $\{A_{kj}\}$ is Hausdorff statistically convergent to A, then it is Wijsman statistically convergent to A.

Proof. For any $\{A_{kj}\}$, each $\varepsilon > 0$ and $x \in X$, since

$$|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}|$$
$$\leq \left|\left\{k \leq m, j \leq n : \sup_{x \in X} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|,$$

then we have the result. □

Definition 3.12. The sequence $\{A_{kj}\}$ is Hausdorff statistically Cauchy if for each $\varepsilon > 0$ there exist numbers $r = r(\varepsilon), s = s(\varepsilon) \in \mathbb{N}$ such that

$$\lim_{m,n \to \infty} \frac{1}{mn}|\{k \leq m, j \leq n : \sup_{x \in X} |d(x, A_{kj}) - d(x, A_{rs})| \geq \varepsilon\}| = 0.$$

Theorem 3.13. A double sequence $\{A_{kj}\}$ is Hausdorff statistically convergent if and only if $\{A_{kj}\}$ is Hausdorff statistically Cauchy.

Proof. The proof is similar to the proof of Theorem 3.10 □

Acknowledgments

The authors wish to thank the referee for some helpful comments and suggestions.

References

1. B. Altay, F. Başar, Some new spaces of double sequences, *J. Math. Anal. Appl.*, **309** (1), (2005), 70–90.
2. J.-P. Aubin, and H. Frankowska, *Set-valued analysis*, Birkhauser, Boston, 1990.
3. M. Baronti, and P. Papini, Convergence of sequences of sets, *Methods of functional analysis in approximation theory*,**76**, Birkhauser-Verlag, Basel,(1986), 133-155.
4. G. Beer, On convergence of closed sets in a metric space and distance functions, *Bull. Aust. Math. Soc.*, **31**, (1985), 421–432.
5. G. Beer, Wijsman convergence: A survey, *Set-Valued Var. Anal.*, **2**, (1994), 77–94.
6. J. S. Connor, The statistical and strong p-Cesàro convergence of sequences, *Analysis*, **8**, (1988), 46–63.
7. C. çakan, B. Altay, Statistically boundedness and statistical core of double sequences, *J. Math. Anal. Appl.*, **317**, (2006), 690–697.
8. R. Çolak, Y. Altın, Statistical convergence of double sequences of order α, *Journal of Function Spaces and Applications*, **2013**, (2013), 1–5.

9. H. Fast, Sur la convergence statistique, *Colloq. Math.*,**2**, (1951), 241–244.
10. A. R. Freedman, J.J. Sember, M. Raphael, Some Cesàro type summability spaces, *Proc. London Math. Soc.*, **37**, (1978), 508–520.
11. J. A. Fridy, C. Orhan, Statistical limit superior and inferior, *Proc. Amer. Math. Soc.*, **125**, (1997) 3625–3631.
12. J. D. Hill, On perfect summability of double sequences, *Bull. Amer. Math. Soc.*, **46**, (1940), 327-331.
13. M. Işık, Y. Altın, $f_{(\lambda,\mu)}$-statistical convergence of order $\widetilde{\alpha}$ for double sequences, *Journal of Inequalities and Applications*, **2017**(246), (2017), 8 pages.
14. I. G. Kull, Multiplication of summable double series, *Uch. zap. Tartusskogo un-ta*, **62**, (1958), 3–59 (in Russian).
15. B. V. Limayea, M. Zeltser, On the Pringsheim convergence of double series, *Proc. Est. Acad. Sci.*, **58**, (2009), 108–121.
16. M. Mursaleen, O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288**, (2003), 223–231.
17. F. Nuray, B. E. Rhoades, Statistical convergence of sequences of sets, *Fasc. Math.*, **49**, (2012), 87–99.
18. F. Nuray, U. Ulusu, E. Dündar, Cesàro summability of double sequences of sets, *Gen. Math. Notes* **25**(1), (2014), 8–18.
19. A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann.*, **53**, (1900), 289–321.
20. R.T. Rockafellar, R.J-B Wets, Variational Analysis, *Grundlehren der Mathematischen Wissenschaften* 317, Springer-Verlag 2009.
21. E. Savaş, On some double lacunary sequence spaces of fuzzy numbers, *Mathematical and Computational Applications*, **15**(3), (2010), 439–448.
22. I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66**,(1959), 361–375.
23. Y. Sever, Ö. Talo, On Statistical Convergence of Double Sequences of Closed Sets, *Filomat*, **30**(3), (2016), 533–539, DOI 10.2298/FIL1603533S.
24. Y. Sever, Ö. Talo, B. Altay, On convergence of double sequences of closed sets, *Contemp. Anal. Appl. Math.*, **3**, (2015), 30–49.
25. Ö. Talo, Y. Sever, F. Başar, On statistically convergent sequences of closed set, *Filomat*, **30**(6), (2016), 1497–1509.
26. U. Ulusu, F. Nuray, Lacunary statistical convergence of sequence of sets, *Progress in Applied Mathematics*, **4**(2), (2012), 99–109.
27. U. Ulusu, F. Nuray, On strongly lacunary summability of sequences of sets, *Journal of Applied Mathematics and Bioinformatics*, **3**3, (2013), 75–88.
28. R. A. Wijsman, Convergence of sequences of convex sets, cones and functions, *Bull. Amer. Math. Soc.*, **70**, (1964), 186–188.
29. R. A. Wijsman, Convergence of Sequences of Convex sets, Cones and Functions II, *Trans. Amer. Math. Soc.*, **123**(1), (1966), 32–45.
30. M. Zeltser, M. Mursaleen, S. A. Mohiuddine, *On almost conservative matrix methods for double sequence spaces*, Publ. Math. Debrecen, **75** (2009), 1–13.