

Existence and Iterative Approximation of Solution for Generalized Yosida Inclusion Problem

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ABSTRACT. In this paper, we introduce and study a generalized Yosida approximation operator associated to $H(\cdot, \cdot)$ -co-accretive operator and discuss some of its properties. Using the concept of graph convergence and resolvent operator, we establish the convergence for generalized Yosida approximation operator. Also, we show an equivalence between graph convergence for $H(\cdot, \cdot)$ -co-accretive operator and generalized Yosida approximation operator. Furthermore, we suggest an iterative algorithm to solve a Yosida inclusion problem under some mild conditions in q -uniformly smooth Banach space and discuss the convergence and uniqueness of the solution.

Keywords: Graph convergence, Resolvent operator, Iterative algorithm, Yosida approximation operator, Yosida inclusion problem.

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1. INTRODUCTION

The theory of variational inequalities (variational inclusions) is one of the most important and dignified areas in nonlinear analysis and optimization because number of problems from science, engineering, social sciences, management, etc., can be modeled in the form of a variational inequality or variational inclusion; see, for example, [7, 8, 10, 12, 13, 14, 19, 20] and the references

therein. This theory has been generalized by many researchers in variant directions and many efficient methods have been developed in the last three decades to solve variational inequalities or variational inclusions; see, for example, [1, 2, 5, 6, 8, 10, 11, 14, 15, 16, 24, 25, 29]. The notion of monotone operators independently introduced by Zarantonello [28] and Minty [23]. The monotone operators are very interesting to study and researched by number of authors as they have firm connection with the following first order evolution equation:

$$\frac{dx}{dt} = -M(x), \quad x(0) = x_0,$$

which is the model of many problems of physical applications.

It is well known that two quite useful single-valued Lipschitz continuous operators can be associated with a monotone operators, namely resolvent operator and Yosida approximation operator. The Yosida approximation operators are useful to approximate the solutions of variational inclusion problems using non-expansive resolvent operators. Recently, many authors implemented Yosida approximation operators to study some of variational inclusion problems using different approaches; see, for example, [9, 17, 18, 21, 26].

In this paper, motivated by the research discussed above, we introduced a generalized Yosida approximation operator associated to $H(\cdot, \cdot)$ -co-accretive operator and discuss some of its properties. We discuss the convergence of generalized Yosida approximation operator and establish its equivalence with graph convergence for $H(\cdot, \cdot)$ -co-accretive operator in q -uniformly smooth Banach space. Further, we suggest an iterative algorithm and investigate the convergence of iterative algorithm. Also, we solve a Yosida inclusion problem as an application and discuss the existence and uniqueness of the solution. Our results refine and generalize some known results in literature.

2. PRELIMINARIES

Let E be a real Banach space with its norm $\|\cdot\|$, E^* be the topological dual of E and d be the metric induced by the norm $\|\cdot\|$. Let $\langle \cdot, \cdot \rangle$ be the dual pair between E and E^* and $CB(E)$ (respectively 2^E) be the family of all nonempty closed and bounded subsets (respectively, all non empty subsets) of E .

The generalized duality mapping $J_q : E \rightarrow 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \right\}, \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that $J_q(x) = \|x\|^{q-1} J_2(x), \forall x (\neq 0) \in E$. In the sequel, we assume that E is a real Banach space such that J_q is single-valued. If E is a real Hilbert space, then J_2 becomes the identity mapping on E .

The modulus of smoothness of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

A Banach space E is called uniformly smooth, if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0;$$

E is called q -uniformly smooth, if there exists a constant $c > 0$ such that

$$\rho_E(t) \leq ct^q, \quad q > 1.$$

Note that J_q is single-valued, if E is uniformly smooth. Xu [27] proved the following important inequality in q -uniformly smooth Banach spaces.

Lemma 2.1. *Let $q > 1$ be a real number and let E be a real uniformly smooth Banach space. Then E is q -uniformly smooth if and only if there exists a constant $c_q > 0$ such that for every $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q\|y\|^q.$$

Definition 2.2. A mapping $A : E \rightarrow E$ is said to be

(i) accretive, if

$$\langle Ax - Ay, J_q(x - y) \rangle \geq 0, \quad \forall x, y \in E;$$

(ii) strictly accretive, if

$$\langle Ax - Ay, J_q(x - y) \rangle > 0, \quad \forall x, y \in E,$$

and the equality holds if and only if $x = y$;

(iii) δ -strongly accretive, if there exists a constant $\delta > 0$ such that

$$\langle Ax - Ay, J_q(x - y) \rangle \geq \delta\|x - y\|^q, \quad \forall x, y \in E;$$

(iv) β -relaxed accretive, if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, J_q(x - y) \rangle \geq (-\beta)\|x - y\|^q, \quad \forall x, y \in E;$$

(v) μ -cocoercive, if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, J_q(x - y) \rangle \geq \mu\|Ax - Ay\|^q, \quad \forall x, y \in E;$$

(vi) γ -relaxed cocoercive, if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, J_q(x - y) \rangle \geq (-\gamma)\|Ax - Ay\|^q, \quad \forall x, y \in E;$$

(vii) σ -Lipschitz continuous, if there exists a constant $\sigma > 0$ such that

$$\|Ax - Ay\| \leq \sigma\|x - y\|, \quad \forall x, y \in E;$$

(viii) η -expansive, if there exists a constant $\eta > 0$ such that

$$\|Ax - Ay\| \geq \eta\|x - y\|, \quad \forall x, y \in E;$$

if $\eta = 1$, then it is expansive.

Definition 2.3. Let $H : E \times E \rightarrow E$ and $A, B : E \rightarrow E$ be three single-valued mappings. Then

- (i) $H(A, \cdot)$ is said to be μ_1 -cocoercive with respect to A , if there exists a constant $\mu_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \mu_1 \|Ax - Ay\|^q, \quad \forall x, y, u \in E;$$
- (ii) $H(\cdot, B)$ is said to be γ_1 -relaxed cocoercive with respect to B , if there exists a constant $\gamma_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq (-\gamma_1) \|Bx - By\|^q, \quad \forall x, y, u \in E;$$
- (iii) $H(A, B)$ is said to be symmetric cocoercive with respect to A and B , if $H(A, \cdot)$ is cocoercive with respect to A and $H(\cdot, B)$ is relaxed cocoercive with respect to B ;
- (iv) $H(A, \cdot)$ is said to be α_1 -strongly accretive with respect to A , if there exists a constant $\alpha_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \alpha_1 \|x - y\|^q, \quad \forall x, y, u \in E;$$
- (v) $H(\cdot, B)$ is said to be β_1 -relaxed accretive with respect to B , if there exists a constant $\beta_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq (-\beta_1) \|x - y\|^q, \quad \forall x, y, u \in E;$$
- (vi) $H(A, B)$ is said to be symmetric accretive with respect to A and B , if $H(A, \cdot)$ is strongly accretive with respect to A and $H(\cdot, B)$ is relaxed accretive with respect to B ;
- (vii) $H(A, \cdot)$ is said to be ξ_1 -Lipschitz continuous with respect to A , if there exists a constant $\xi_1 > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leq \xi_1 \|x - y\|, \quad \forall x, y, u \in E;$$
- (viii) $H(\cdot, B)$ is said to be ξ_2 -Lipschitz continuous with respect to B , if there exists a constant $\xi_2 > 0$ such that

$$\|H(u, Bx) - H(u, By)\| \leq \xi_2 \|x - y\|, \quad \forall x, y, u \in E.$$

Definition 2.4. Let $f, g : E \rightarrow E$ be two single-valued mappings and $M : E \times E \rightarrow 2^E$ be a multi-valued mapping. Then

- (i) $M(f, \cdot)$ is said to be α -strongly accretive with respect to f , if there exists a constant $\alpha > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq \alpha \|x - y\|^q, \quad \forall x, y, w \in E \quad \text{and} \quad \forall u \in M(f(x), w), \\ v \in M(f(y), w);$$
- (ii) $M(\cdot, g)$ is said to be β -relaxed accretive with respect to g , if there exists a constant $\beta > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq (-\beta) \|x - y\|^q, \quad \forall x, y, w \in E \quad \text{and} \quad \forall u \in M(w, g(x)), \\ v \in M(w, g(y));$$
- (iii) $M(f, g)$ is said to be symmetric accretive with respect to f and g , if $M(f, \cdot)$ is strongly accretive with respect to f and $M(\cdot, g)$ is relaxed accretive with respect to g .

Definition 2.5. Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings. Let $M : E \times E \rightarrow 2^E$ be a multi-valued mapping. The

mapping M is said to be $H(\cdot, \cdot)$ -co-accretive with respect to A, B, f and g , if $H(A, B)$ is symmetric cocoercive with respect to A and B , $M(f, g)$ is symmetric accretive with respect to f and g and $(H(A, B) + \lambda M(f, g))(E) = E$, for every $\lambda > 0$.

Theorem 2.6. [4] *Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings. Let $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . Let A be η -expansive and B be σ -Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the mapping $[H(A, B) + \lambda M(f, g)]^{-1}$ is single-valued, for all $\lambda > 0$.*

Definition 2.7. [4] *Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings. Let $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . The resolvent operator $R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)} : E \rightarrow E$ is defined by*

$$R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) = [H(A, B) + \lambda M(f, g)]^{-1}(u), \quad \forall u \in E, \lambda > 0. \quad (2.1)$$

Lemma 2.8. *Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings. Let $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . Then the resolvent operator $R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ defined in (2.1) is θ -Lipschitz continuous, where*

$$\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)}.$$

3. CONVERGENCE OF GENERALIZED YOSIDA APPROXIMATION OPERATOR

In this section, we define the generalized Yosida approximation operator by using the concept of resolvent operator and discuss its convergence.

Definition 3.1. The generalized Yosida approximation operator $J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)} : E \rightarrow E$ is defined as

$$J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) = \frac{1}{\lambda} \left[I - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)} \right](u), \quad \forall u \in E, \lambda > 0, \quad (3.1)$$

where, I is the identity mapping on E .

Lemma 3.2. *Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings. Suppose $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . Let A be η -expansive and B be σ -Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the generalized Yosida approximation operator $J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ defined in (3.1) is*

- (i) m_1 -Lipschitz continuous,
- (ii) m_2 -strongly monotone,

where, $m_1 = \frac{1}{\lambda}(1 + \theta)$, $m_2 = \frac{1}{\lambda}(1 - \theta)$ and $\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)}$.

Proof. (i). Let u, v be any given points in E . It follows from the definition of generalized Yosida approximation operator and Lipschitz continuity of resolvent operator that

$$\begin{aligned} \|J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)\| &= \frac{1}{\lambda} \|[I(u) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u)] - [I(v) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)]\| \\ &\leq \frac{1}{\lambda} [\|u - v\| + \|R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)\|] \\ &\leq \frac{1}{\lambda} [\|u - v\| + \theta \|u - v\|] \\ &= \frac{1}{\lambda} (1 + \theta) \|u - v\|. \end{aligned}$$

That is,

$$\|J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)\| \leq m_1 \|u - v\|,$$

where, $m_1 = \frac{1}{\lambda}(1 + \theta)$ and $\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)}$.

(ii) Let u, v be any given points in E , then again using the definition of generalized Yosida approximation operator, we get

$$\begin{aligned} \langle J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v), J_q(u - v) \rangle &= \frac{1}{\lambda} \langle (I(u) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u)) \\ &\quad - (I(v) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)), J_q(u - v) \rangle \\ &= \frac{1}{\lambda} [\langle u - v, J_q(u - v) \rangle - \langle R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) \\ &\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v), J_q(u - v) \rangle] \\ &\geq \frac{1}{\lambda} [\|u - v\|^q - \|R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)\| \|u - v\|^{q-1}]. \end{aligned}$$

Now using the Lipschitz continuity of resolvent operator, we have

$$\begin{aligned} \langle J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v), J_q(u - v) \rangle &\geq \frac{1}{\lambda} [\|u - v\|^q - \theta \|u - v\|^q] \\ &= \frac{1}{\lambda} (1 - \theta) \|u - v\|^q. \end{aligned}$$

That is,

$$\langle J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v), J_q(u - v) \rangle \geq m_2 \|u - v\|^q,$$

where, $m_2 = \frac{1}{\lambda}(1 - \theta)$ and $\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)}$. This completes the proof. \square

Definition 3.3. [4] Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings. Let $M_n, M : E \times E \rightarrow 2^E$ be $H(\cdot, \cdot)$ -co-accretive mappings, for $n = 0, 1, 2, \dots$. The sequence $\{M_n\}$ is said to be graph convergence to M , denoted by $M_n \xrightarrow{G} M$, if for every $((f(x), g(x)), z) \in \text{graph}(M)$, there exists a sequence $((f(x_n), g(x_n)), z_n) \in \text{graph}(M_n)$ such that

$$f(x_n) \rightarrow f(x), g(x_n) \rightarrow g(x) \text{ and } z_n \rightarrow z \text{ as } n \rightarrow \infty.$$

Theorem 3.4. [4] *Let $A, B, f, g : E \rightarrow E$ be the single-valued mappings and $M_n, M : E \times E \rightarrow 2^E$ be $H(\cdot, \cdot)$ -co-accretive mappings with respect to A, B, f and g . Let $H : E \times E \rightarrow E$ be a single-valued mapping such that*

- (i) $H(A, B)$ is ξ_1 -Lipschitz continuous with respect to A and ξ_2 -Lipschitz continuous with respect to B ;
- (ii) f is a continuous τ -expansive mapping.

Then $M_n \underline{G} M$ if and only if

$$R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) \rightarrow R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u), \forall u \in E, \lambda > 0,$$

where,

$$R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) = [H(A, B) + \lambda M_n(f, g)]^{-1}(u) \text{ and } R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) = [H(A, B) + \lambda M(f, g)]^{-1}(u).$$

Next, we prove the convergence of generalized Yosida approximation operator by using the concept of graph convergence for $H(\cdot, \cdot)$ -co-accretive operator.

Theorem 3.5. *Let $A, B, f, g : E \rightarrow E$ be the single-valued mappings and $M_n, M : E \times E \rightarrow 2^E$ be $H(\cdot, \cdot)$ -co-accretive mappings with respect to A, B, f and g . Let $H : E \times E \rightarrow E$ be a single-valued mapping such that the conditions (i) and (ii) of the Theorem 3.4 hold. Then $M_n \underline{G} M$ if and only if*

$$J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) \rightarrow J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x), \forall x \in E, \lambda > 0,$$

where,

$$J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) = \frac{1}{\lambda} [I - R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}](x), J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) = \frac{1}{\lambda} [I - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}](x),$$

and $R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}, R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ are same as defined in Theorem 3.4.

Proof. Suppose that $M_n \underline{G} M$, then for any given $x \in E$, let

$$z_n = J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) \text{ and } z = J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x).$$

Then $z = J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) = \frac{1}{\lambda} [I - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}](x)$, thus

$$x - \lambda z = R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) = [H(A, B) + \lambda M(f, g)]^{-1}(x),$$

which implies that

$$H(A, B)(x - \lambda z) + \lambda M(f, g)(x - \lambda z) = x$$

$$\text{i.e., } \frac{1}{\lambda} [x - H(A, B)(x - \lambda z)] \in M(f(x - \lambda z), g(x - \lambda z)).$$

Thus, we have

$$((f(x - \lambda z), g(x - \lambda z)), \frac{1}{\lambda} (x - H(A(x - \lambda z), B(x - \lambda z)))) \in \text{graph}(M).$$

Then by the definition of graph convergence, there exists a sequence $\{(f(z'_n), g(z'_n)), y'_n\} \in \text{graph}(M_n)$ such that

$$f(z'_n) \rightarrow f(x-\lambda z), g(z'_n) \rightarrow g(x-\lambda z) \text{ and } y'_n \rightarrow \frac{1}{\lambda}(x-H(A(x-\lambda z), B(x-\lambda z))) \text{ as } n \rightarrow \infty. \quad (3.2)$$

Since $y'_n \in M_n(f(z'_n), g(z'_n))$, we have

$$H(A(z'_n), B(z'_n)) + \lambda y'_n \in [H(A, B) + \lambda M_n(f, g)](z'_n)$$

and thus,

$$\begin{aligned} z'_n &= R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(z'_n), B(z'_n)) + \lambda y'_n] \\ &= (I - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)})[H(A(z'_n), B(z'_n)) + \lambda y'_n], \end{aligned}$$

which implies that

$$\frac{1}{\lambda}z'_n = \frac{1}{\lambda}H(A(z'_n), B(z'_n)) + y'_n - J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(z'_n), B(z'_n)) + \lambda y'_n].$$

Now,

$$\begin{aligned} \|z_n - z\| &= \|J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - z\| \\ &= \|J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) + \frac{1}{\lambda}z'_n - \frac{1}{\lambda}z'_n - z\| \\ &= \|J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) + \frac{1}{\lambda}(H(A(z'_n), B(z'_n))) + y'_n \\ &\quad - J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(z'_n), B(z'_n)) + \lambda y'_n] - \frac{1}{\lambda}z'_n - z\| \\ &\leq \|J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(z'_n), B(z'_n)) + \lambda y'_n]\| \\ &\quad + \|\frac{1}{\lambda}H(A(z'_n), B(z'_n)) + y'_n - \frac{1}{\lambda}z'_n - z\|. \end{aligned}$$

Using the Lipschitz continuity of generalized Yosida approximation operator, we get

$$\begin{aligned} \|z_n - z\| &\leq m_1 \|x - H(A(z'_n), B(z'_n)) - \lambda y'_n\| + \|\frac{1}{\lambda}H(A(z'_n), B(z'_n)) + y'_n - \frac{1}{\lambda}x\| \\ &\quad + \|\frac{1}{\lambda}z'_n - \frac{1}{\lambda}x + z\| \\ &= (m_1 + \frac{1}{\lambda}) \|x - H(A(z'_n), B(z'_n)) - \lambda y'_n\| + \frac{1}{\lambda} \|z'_n - x + \lambda z\| \\ &= (m_1 + \frac{1}{\lambda}) \|x - H(A(z'_n), B(z'_n)) + H(A, B)(x - \lambda z) \\ &\quad - H(A, B)(x - \lambda z) - \lambda y'_n\| + \frac{1}{\lambda} \|z'_n - x + \lambda z\| \\ &\leq (m_1 + \frac{1}{\lambda}) \|x - H(A, B)(x - \lambda z) - \lambda y'_n\| + (m_1 + \frac{1}{\lambda}) \|H(A, B)(x - \lambda z) \\ &\quad - H(A(z'_n), B(z'_n))\| + \frac{1}{\lambda} \|z'_n - x + \lambda z\|. \end{aligned} \quad (3.3)$$

Since H is ξ_1 -Lipschitz continuous with respect to A and ξ_2 -Lipschitz continuous with respect to B , we have

$$\begin{aligned} & \|H(A, B)(x - \lambda z) - H(A(z'_n), B(z'_n))\| \\ &= \|H(A(x - \lambda z), B(x - \lambda z)) - H(A(z'_n), B(z'_n))\| \\ &= \|H(A(x - \lambda z), B(x - \lambda z)) - H(A(x - \lambda z), B(z'_n))\| \\ &\quad + \|H(A(x - \lambda z), B(z'_n)) - H(A(z'_n), B(z'_n))\| \\ &\leq (\xi_1 + \xi_2)\|x - \lambda z - z'_n\|. \end{aligned} \tag{3.4}$$

Thus, it follows from (3.3) and (3.4) that

$$\begin{aligned} \|z_n - z\| &\leq (m_1 + \frac{1}{\lambda})\|x - H(A(x - \lambda z), B(x - \lambda z)) - \lambda y'_n\| \\ &\quad + [(m_1 + \frac{1}{\lambda})(\xi_1 + \xi_2) + \frac{1}{\lambda}]\|x - \lambda z - z'_n\|. \end{aligned} \tag{3.5}$$

Since f is τ -expansive, we have

$$\|f(z'_n) - f(x - \lambda z)\| \geq \tau\|z'_n - (x - \lambda z)\| \geq 0. \tag{3.6}$$

Since $f(z'_n) \rightarrow f(x - \lambda z)$ as $n \rightarrow \infty$. By (3.6), we have $z'_n \rightarrow x - \lambda z$ as $n \rightarrow \infty$. Also from (3.2), we have $y'_n \rightarrow \frac{1}{\lambda}(x - H(A(x - \lambda z), B(x - \lambda z)))$ as $n \rightarrow \infty$. Thus, it follows from (3.5) that

$$\|z_n - z\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which implies that

$$J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) \rightarrow J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x).$$

Conversely, suppose that

$$J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) \rightarrow J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x), \quad \forall x \in E, \lambda > 0.$$

For any $((f(x), g(x)), y) \in \text{graph}(M)$, we have

$$y \in M(f(x), g(x)),$$

thus,

$$H(A(x), B(x)) + \lambda y \in [H(A, B) + \lambda M(f, g)](x)$$

and so

$$x = R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) + \lambda y] = [I - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}][H(A(x), B(x)) + \lambda y].$$

Let $x_n = [I - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}][H(A(x), B(x)) + \lambda y]$, then

$$\frac{1}{\lambda}[H(A(x), B(x)) - H(A(x_n), B(x_n)) + \lambda y] \in M_n(f(x_n), g(x_n)).$$

Let $y'_n = \frac{1}{\lambda}[H(A(x), B(x)) - H(A(x_n), B(x_n)) + \lambda y]$.

Now,

$$\begin{aligned} \|y'_n - y\| &= \|\frac{1}{\lambda}[H(A(x), B(x)) - H(A(x_n), B(x_n)) + \lambda y] - y\| \\ &= \frac{1}{\lambda}\|H(A(x), B(x)) - H(A(x_n), B(x_n))\|. \end{aligned}$$

Then by using the same argument as in (3.4), we get

$$\|y'_n - y\| \leq \frac{(\xi_1 + \xi_2)}{\lambda} \|x_n - x\|. \quad (3.7)$$

Also,

$$\begin{aligned} \|x_n - x\| &= \|[I - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}][H(A(x), B(x)) + \lambda y] \\ &\quad - [I - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}][H(A(x), B(x)) + \lambda y]\| \\ &= \lambda \|[J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)} - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}][H(A(x), B(x)) + \lambda y]\|. \end{aligned}$$

Since $J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)} \rightarrow J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$, we have $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Thus from (3.7), we have $\|y'_n - y\| \rightarrow 0$ as $n \rightarrow \infty$. Hence by the continuity of mapping f , $M_n \underline{G} M$. This completes the proof. \square

Remark 3.6. The convergence of the resolvent operator $R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)} \rightarrow R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ and the convergence of generalized Yosida approximation operator $J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)} \rightarrow J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ are equivalent, if and only if $M_n \underline{G} M$.

4. A YOSIDA INCLUSION PROBLEM AND EXISTENCE RESULT

Let E be a q -uniformly smooth Banach space with norm $\|\cdot\|$. Let $A, B, f, g : E \rightarrow E; H : E \times E \rightarrow E$ be the single-valued mappings and $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . We consider the following Yosida inclusion problem (in short: YIP):

Find $x \in E$ such that

$$0 \in J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) + M(f(x), g(x)), \quad (4.1)$$

where $J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ is generalized Yosida approximation operator.

Remark 4.1. For suitable choices of the mappings involved in the formulation of YIP (4.1), one can obtain many problems existing in literature; see, [3, 4, 22].

Lemma 4.2. *Let $A, B, f, g : E \rightarrow E; H : E \times E \rightarrow E$ be the single-valued mappings and $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . Then $x \in E$ is a solution of YIP (4.1), if and only if x satisfies the following equation:*

$$x = R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]. \quad (4.2)$$

Proof. The proof of the lemma follows directly from the definition of resolvent operator $R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$.

Next, we suggest the following iterative algorithm for finding an approximate solution for YIP (4.1).

Algorithm 4.3. For any $x \in E$, compute the sequence $\{x_n\} \subset E$ by the following iterative scheme:

$$x_{n+1} = R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)], \quad (4.3)$$

where $n = 0, 1, 2, \dots$; $\lambda > 0$ is a constant.

Theorem 4.4. Let E be a q -uniformly smooth Banach space with norm $\|\cdot\|$ and $A, B, f, g : E \rightarrow E$ be the single-valued mappings such that A is η -expansive and B is σ -Lipschitz continuous. Let $H : E \times E \rightarrow E$ be a symmetric cocoercive mapping with respect to A and B with constants μ and γ respectively, ξ_1 -Lipschitz continuous with respect to A and ξ_2 -Lipschitz continuous with respect to B . Let $M : E \times E \rightarrow 2^E$ be an $H(\cdot, \cdot)$ -co-accretive mapping with respect to A, B, f and g . Suppose that there exists a constant $\lambda > 0$ satisfying

$$0 < \theta \left[\sqrt[q]{1 - 2q(\mu\eta^q - \gamma\sigma^q) + c_q(\xi_1 + \xi_2)^q} + \sqrt[q]{1 - 2\lambda q m_1 + c_q \lambda^q m_2^q} \right] < 1, \quad (4.4)$$

where, $\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)}$, $m_1 = \frac{1}{\lambda}(1 + \theta)$, $m_2 = \frac{1}{\lambda}(1 - \theta)$, $\alpha > \beta, \mu > \gamma, \eta > \sigma$.

Then YIP (4.1) has a unique solution.

Proof. We define a mapping $T : E \rightarrow E$ by

$$T(x) = R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)], \quad \forall x \in E, \lambda > 0. \quad (4.5)$$

For any $x, y \in E$, using (4.5) and Lipschitz continuity of $R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$, we have

$$\begin{aligned} \|T(x) - T(y)\| &= \|R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)] \\ &\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(y), B(y)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y)]\| \\ &\leq \theta \| [H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)] \\ &\quad - [H(A(y), B(y)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y)] \| \\ &\leq \theta \| H(A(x), B(x)) - H(A(y), B(y)) - (x - y) \| \\ &\quad + \theta \| (x - y) - \lambda (J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y)) \|. \end{aligned} \quad (4.6)$$

Since H is symmetric cocoercive with respect to A and B with constants μ and γ , respectively, ξ_1 -Lipschitz continuous with respect to A and ξ_2 -Lipschitz continuous with respect to B , then using Lemma 2.1, we have

$$\begin{aligned} &\|H(A(x), B(x)) - H(A(y), B(y)) - (x - y)\|^q \\ &\leq \|x - y\|^q - q \langle H(A(x), B(x)) - H(A(y), B(y)), \\ &\quad J_q(x - y) \rangle + c_q \|H(A(x), B(x)) - H(A(y), B(y))\|^q \\ &\leq \|x - y\|^q - q(\mu \|A(x) - A(y)\|^q \\ &\quad - \gamma \|B(x) - B(y)\|^q) + c_q(\xi_1 + \xi_2)^q \|x - y\|^q. \end{aligned} \quad (4.7)$$

Since A is η -expansive and B is σ -Lipschitz continuous, we have

$$\begin{aligned} & \|H(A(x), B(x)) - H(A(y), B(y)) - (x - y)\|^q \\ & \leq [1 - q(\mu\eta^q - \gamma\sigma^q) + c_q(\xi_1 + \xi_2)^q] \|x - y\|^q, \end{aligned}$$

which implies that

$$\begin{aligned} & \|H(A(x), B(x)) - H(A(y), B(y)) - (x - y)\| \\ & \leq \sqrt[q]{1 - q(\mu\eta^q - \gamma\sigma^q) + c_q(\xi_1 + \xi_2)^q} \|x - y\|. \end{aligned} \quad (4.8)$$

Since $J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ is m_1 -Lipschitz continuous and m_2 -strongly monotone, then we have

$$\begin{aligned} & \|(x - y) - \lambda(J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y))\|^q \\ & \leq \|x - y\|^q - \lambda q \langle J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y), J_q(x - y) \rangle \\ & \quad + c_q \lambda^q \|J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y)\|^q \\ & \leq (1 - \lambda q m_1 + c_q \lambda^q m_2^q) \|x - y\|^q, \end{aligned}$$

which implies that

$$\|(x - y) - \lambda(J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(y))\| \leq \sqrt[q]{1 - \lambda q m_1 + c_q \lambda^q m_2^q} \|x - y\|. \quad (4.9)$$

From equations (4.8), (4.9) and (4.6), we have

$$\begin{aligned} \|T(x) - T(y)\| & \leq \theta \left[\sqrt[q]{1 - q(\mu\eta^q - \gamma\sigma^q) + c_q(\xi_1 + \xi_2)^q} \right. \\ & \quad \left. + \sqrt[q]{1 - \lambda q m_1 + c_q \lambda^q m_2^q} \right] \|x - y\|. \end{aligned}$$

That is,

$$\|T(x) - T(y)\| \leq L \|x - y\|, \quad (4.10)$$

where, $L = \theta \left[\sqrt[q]{1 - q(\mu\eta^q - \gamma\sigma^q) + c_q(\xi_1 + \xi_2)^q} + \sqrt[q]{1 - \lambda q m_1 + c_q \lambda^q m_2^q} \right]$.

Since $0 < L < 1$ by condition (4.4), it follows from (4.10) that T is a contraction mapping. Thus, the mapping T has a unique fixed point $x \in E$. Hence $x \in E$ is the unique solution of YIP (4.1).

Theorem 4.5. *Let E be a q -uniformly smooth Banach space with norm $\|\cdot\|$. Let $A, B, f, g : E \rightarrow E$ and $H : E \times E \rightarrow E$ be the single-valued mappings such that H is ξ_1 -Lipschitz continuous and ξ_2 -Lipschitz continuous with respect to B . Let $M_n, M : E \times E \rightarrow 2^E$ be $H(\cdot, \cdot)$ -co-accretive mappings such that $M_n \xrightarrow{\mathcal{G}} M$. In addition the following condition is satisfied:*

$$0 < \theta[\xi_1 + \xi_2 + \lambda m_1] < 1. \quad (4.11)$$

Then the approximate solution $\{x_n\}$ generated by Algorithm 4.3 converges strongly to the unique solution x of YIP (4.1).

Proof. It follows from Algorithm 4.3 and Lipschitz continuity of resolvent operator that

$$\begin{aligned}
\|x_{n+1} - x\| &= \|R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]\| \\
&= \|R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad + R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]\| \\
&\leq \|R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)]\| \\
&\quad + \|R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]\| \\
&\leq a_n + \theta \|H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n) \\
&\quad - [H(A(x), B(x)) - \lambda J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]\|,
\end{aligned} \tag{4.12}$$

where,

$$\begin{aligned}
a_n &= \|R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)] \\
&\quad - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}[H(A(x_n), B(x_n)) - \lambda J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)]\|.
\end{aligned} \tag{4.13}$$

It follows from the Lipschitz continuity of H and Lipschitz continuity of generalized Yosida approximation operator that

$$\begin{aligned}
&\|H(A(x_n), B(x_n)) - H(A(x), B(x)) - \lambda[J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]\|, \\
&\leq \|H(A(x_n), B(x_n)) - H(A(x_n), B(x)) \\
&\quad + H(A(x_n), B(x)) - H(A(x), B(x)) \\
&\quad - \lambda[J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)]\| \\
&\leq \|H(A(x_n), B(x_n)) - H(A(x_n), B(x))\| \\
&\quad + \|H(A(x_n), B(x)) - H(A(x), B(x))\| \\
&\quad + \lambda \|J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n)\| \\
&\quad + \lambda \|J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)\| \\
&\leq \xi_2 \|x_n - x\| + \xi_1 \|x_n - x\| + \lambda b_n + \lambda m_1 \|x_n - x\|,
\end{aligned} \tag{4.14}$$

where,

$$b_n = \|J_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(x_n) - J_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(x)\|. \tag{4.15}$$

Then from (4.12) and (4.14), we obtain

$$\|x_{n+1} - x\| \leq a_n + \theta[\xi_1 + \xi_2 + \lambda m_1]\|x_n - x\| + \theta \lambda b_n.$$

Thus, we have

$$\|x_{n+1} - x\| \leq M\|x_n - x\| + a_n + \theta\lambda b_n, \quad (4.16)$$

where

$$M = \theta[\xi_1 + \xi_2 + \lambda m_1].$$

From (4.11), we have $0 < M < 1$ and from (4.13) and (4.15), $a_n, b_n \rightarrow 0$ as $n \rightarrow \infty$. Hence (4.16) implies that

$$\|x_{n+1} - x\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, $\{x_n\}$ converges strongly to the unique solution x of YIP (4.1). This completes the proof.

5. CONFLICT OF INTERESTS

The author declares that there is no conflict of interests regarding the publication of this article.

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REFERENCES

1. R. P. Agrawal, R. U. Verma, Generalized system of (A, η) -maximal relaxed monotone variational inclusion problems based on generalized hybrid algorithms, *Commun. Nonlinear Sci. Numer. Simul.* **15**(2), (2010), 238–251.
2. R. Ahmad, B.S Lee, M. Akram, $H(\cdot, \cdot)$ -co-accretive mapping with an application for solving a system of variational inclusions, *Thai J. Math.* **11**(2), (2013), 411–427.
3. R. Ahmad, M. Ishtyak, M. Rahaman, I. Ahmad, Graph convergence and generalized Yosida approximation operator with an application, *Math Sci.* **11**(2), (2017), 155–163.
4. R. Ahmad, M. Akram, M. Dilshad, Graph Convergence for the $H(\cdot, \cdot)$ -co-accretive mapping with an application, *Bull. Malays. Math. Sci. Soc.* **38**(4), (2015), 1481–1506.
5. M. Akram, M. Dilshad, New generalized system of variational inclusions involving $H(\cdot, \cdot)$ -co-accretive mapping, *Afr. Mat.* **29**, (2018), 1173–1187.
6. M. Alansari, M. Akram, M. Dilshad, Iterative algorithms for a generalized system of mixed variational-like inclusion problems and altering points problem, *Stat., Optim. Inf. Comput.* **8**, (2020), 0–15.
7. Q. H. Ansari (ed.), *Topics in Nonlinear Analysis and Optimization*, World Education, Delhi, 2012.
8. Q. H. Ansari, C. S. Lalitha, M. Mehta, *Generalized Convexity, Nonsmooth Variational Inequalities and Nonsmooth Optimization*, CRC Press, Taylor & Francis Group, Boca Raton, London, New York, 2014.
9. H.W. Cao, Yosida approximation equations technique for system of generalized set-valued variational inclusions, *J. Inequal. Appl.* DOI: 10.1186/1029-242X-2013-455 (2013).
10. F. Facchinei, J.-S. Pang, *Finite Dimensional Variational Inequalities and Complementarity Problems*, Volume I and II, Springer-Verlag, New York, Berlin, Heidelberg, 2003.
11. Y. P. Fang, N. J. Huang, H -Monotone operator and resolvent operator technique for variational inclusions, *Appl. Math. Comput.* **145**, (2003), 795–803.

12. F. Giannessi, A. Maugeri (eds.), *Variational Inequalities and Network Equilibrium Problems*, Springer, New York, 1995.
13. F. Giannessi, A. Maugeri (eds.), *Variational Analysis and Applications*, Springer, New York, 2005.
14. R. Glowinski, J. L. Lions, R. Tremolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.
15. N. J. Huang, Y. P. Fang, A new class of general variational inclusions involving maximal η -monotone mappings, *Publ. Math. Debrecen* **62**, (2003), 83–98.
16. Z. Huang, M.A. Noor, Equivalency of convergence between one-step iteration algorithm and two-step iteration algorithm of variational inclusions for H -monotone mappings, *Comput. Math. Appl.* **53**, (2007), 1567–1571.
17. R. Ikehata, N. Okozawa, Yosida approximation and nonlinear hybrid equation, *Nonlinear Anal.* **15**(5), (1990), 479–95.
18. T. Iwamiya, H. Okochi, Monotonicity, resolvents and Yosida approximations of operators on Hilbert manifolds, *Nonlinear Anal.* **54**(2), (2003), 205–214.
19. D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
20. I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, in *Lecture Notes in Mathematical Economics*, Springer-Verlag, Berlin, 2000.
21. H. Y. Lan, Generalized Yosida approximations based on relatively A -maximal m -relaxed monotonicity frameworks, *Abstr. Appl. Anal.* (2013), (article ID 157190).
22. X. Li, N. J. Huang, Graph convergence for the $H(\cdot, \cdot)$ -accretive operator in Banach spaces with an application, *Appl. Math. Comput.* **217**, (2011), 9053–9061.
23. G. J. Minty, “Monotone (nonlinear) operators in Hilbert space,” *Duke Math. J.* **29**, (1962), 341–346.
24. A. Moudafi, A Duality algorithm for solving general variational inclusions, *Adv. Model. Optim.* **13**(2), (2011), 213–220.
25. M. A. Noor, Z. Huang, Some resolvent iterative methods for variational inclusions and non-expansive mappings, *Appl. Math. Comput.* **194**, (2007), 267–275.
26. J.-P. Penot, R. Ratsimahalo, On the Yosida approximation of operators, *Proc. R. Soc. Edinb. Math.* **131A**, (2001), 945–966.
27. H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.* **16**(12), (1991), 1127–1138.
28. E. H. Zarantonello, *Solving functional equations by contractive averaging*, Mathematics Research Center Report 160, Mathematics Research Center, United States Army, University of Wisconsin, 1960.
29. L. C. Zeng, S. M. Guu, J. C. Yao, Characterization of H -monotone operators with applications to variational inclusions, *Comput. Math. Appl.* **50**, (2005), 329–337.