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A Note on Belief Structures and S-approximation Spaces

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Abstract. We study relations between evidence theory and S-approximation spaces. Both theories have their roots in the analysis of Dempster's multivalued mappings and lower and upper probabilities, and have close relations to rough sets. We show that an S-approximation space, satisfying a monotonicity condition, can induce a natural belief structure which is a fundamental block in evidence theory. We also demonstrate that one can induce a natural belief structure on one set, given a belief structure on another set, if the two sets are related by a partial monotone S-approximation space.

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1. Introduction

Dempster-Shafer Theory of Evidence. The Dempster-Shafer theory of evidence is a well-known method in dealing with uncertainty in problems. It originated in 1967 with the introduction of lower and upper probabilities by Dempster [2]. A *belief structure* is a fundamental concept in this theory which assigns two numeric values to each subset of a given set. These values are known as the *belief* and *plausibility* measures. See [8] for a detailed treatment.

S-Approximation. S-approximation spaces are a new way of handling uncertainty, which also originated from Dempster's concepts of lower and upper probabilities [4]. The motivation for this new approach is that it can be seen as a unifying view to rough sets and their extensions, such as [1, 6, 7, 13, 15, 16, 20], since they are all expressible in terms of S-approximation spaces [12, 4]. Hence, any results obtained over S-approximations can be naturally applied to rough sets and many of their extensions, too[‡]. However, S-approximations are capable of representing more than (extensions of) rough sets and model a very broad range of possible approximations (See [4, 12] for more examples).

Previous Works on S-Approximations. The concept of S-approximation has been studied by several approaches and its relation to various theories have been examined. For example, S-approximations are studied in the context of Yao's three-way decisions theory [18, 10] and extended its results. Moreover, they have also been studied in the contexts of neighborhood systems [17, 9], intuitionistic fuzzy set theory [11] and with relations to topology [3].

Motivation. Given the common background and overlap of goals, connections between evidence theory and other theories of approximation have been studied for a long time, e.g. its connections to the theory of rough sets are considered in [5, 14, 19]. The close links between S-approximation spaces and rough sets suggest that a study of relations between evidence theory and S-approximation spaces can yield to more general variants of these results. In this work, we obtain such results about the connections between evidence theory and S-approximation spaces and propose paths for future research.

Organization. The paper is organized as follows: In Section 2, we first review some basic facts from evidence theory, S-approximation spaces, and their corresponding three-way decisions. Then, we study the connection between S-approximation spaces and evidence theory in Section 3. Finally, the paper concludes in Section 4 by suggesting interesting directions for future research.

[‡]This includes all the results reported in the current paper.

2. Preliminaries

2.1. **Dempster-Shafer Theory of Evidence.** In this section, we briefly discuss some background on the Dempster-Shafer theory of evidence. We follow the standard presentation in [8].

Basic Probability Assignments. A *basic probability assignment*, or *bpa* for short, is a fundamental concept in evidence theory. Let W be a finite non-empty set. Then, a bpa over W is a mapping $m : \mathcal{P}(W) \to [0,1]$ satisfying the following conditions: (a) $m(\emptyset) = 0$, and (b) $\sum_{X \subseteq W} m(X) = 1$.

Belief Structures. A set $X \subseteq W$ is called a *focal element* of m if $m(X) \neq 0$. Let $\mathcal{M}$ be the collection of all focal elements of m, then the pair $(\mathcal{M}, m)$ is called a *belief structure* on W.

Belief and Plausibility. Given a belief structure $(\mathcal{M}, m)$, a *belief function* $\mathrm{Bel} : \mathcal{P}(W) \to [0,1]$ and a *plausibility function* $\mathrm{Pl} : \mathcal{P}(W) \to [0,1]$ can be derived, which are defined as follows for every $X \subseteq W$:

$$\mathrm{Bel}(X) := \sum_{Y \subseteq X} m(Y), \tag{2.1}$$

and

$$\mathrm{Pl}(X) := \sum_{Y \cap X \neq \emptyset} m(Y), \tag{2.2}$$

respectively. Note that the Bel and Pl functions are duals, i.e. $\mathrm{Bel}(X) = 1 - \mathrm{Pl}(X^c)$. Moreover, $[\mathrm{Bel}(X), \mathrm{Pl}(X)]$ and $\mathrm{Pl}(X) - \mathrm{Bel}(X)$ are called the *confidence interval* and the *ignorance level* of X, respectively.

Axiomatic Approach. A belief function can equivalently be defined in an axiomatic manner, i.e. it must satisfy the following axioms:

- $\mathrm{Bel}(\emptyset) = 0$,
- $\mathrm{Bel}(W) = 1$,
- $\mathrm{Bel}\left(\cup_{i=1}^{\ell} X_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,\ell\}} (-1)^{|I|+1} \mathrm{Bel}(\cap_{i \in I} X_i)$ for $\{X_1, \ldots, X_\ell\} \subseteq \mathcal{P}(W)$ and $\ell > 0$.

2.2. **S-approximation spaces.** In this section, some basic facts and definitions for S-approximation spaces are presented. We follow the notation of [10, 4].

S-Approximation Spaces. An *S-approximation space* is formally defined as a quadruple $G = (U, W, T, S)$, where U and W are finite non-empty sets, T is a multi-valued mapping $T : U \to \mathcal{P}(W)$, called a *knowledge component*, and S is a mapping $S : \mathcal{P}(W) \times \mathcal{P}(W) \to \{0,1\}$, called a *decider*.

Lower and Upper Approximations. Given an S-approximation space $G = (U, W, T, S)$, the lower and upper approximations of $X \subseteq W$ are defined as

$$\underline{G}(X) = \{x \in U \mid S(T(x), X) = 1\}, \tag{2.3}$$

and

$$\overline{G}(X) = \{x \in U \mid S(T(x), X^c) = 0\}, \tag{2.4}$$

respectively, where X^c denotes the complement of X with respect to W.

Generality and Special Cases. Note that the mapping S can model a large class of measures, of which set inclusion, i.e. $S_{\subseteq}(A,B) = \begin{cases} 1 & A \subseteq B \\ 0 & \text{otherwsie} \end{cases}$, is a special case. If we set S to $S_{\subseteq}$ and consider the sets of form $T(x)$ as blocks, we can model rough sets and some of their generalizations as special cases. For more information and other examples of decider functions consult [4, 10, 9, 12]. Moreover, other definitions and extensions have also been proposed for decider mappings, e.g. refer to [11] to see an instance suitable for intuitionistic fuzzy sets. However, in this paper we stick to the standard and general definition of S-approximation spaces as defined above.

Trichotomy Regions. For any set $X \subseteq W$, the three pair-wise disjoint sets of positive, negative and boundary regions are defined as follows:

$$\begin{aligned}
\mathrm{POS}_G(X) :=& \ \{x \in U \mid S(T(x), X) = 1 \wedge S(T(x), X^c) = 0\} && \text{(Positive Region)}\\
=& \ \underline{G}(X) \cap \overline{G}(X), \\
\mathrm{NEG}_G(X) :=& \ \{x \in U \mid S(T(x), X) = 0 \wedge S(T(x), X^c) = 1\} && \text{(Negative Region)}\\
=& \ U \setminus \left(\underline{G}(X) \cup \overline{G}(X)\right), \\
\mathrm{BR}_G(X) :=& \ \{x \in U \mid S(T(x), X) = S(T(x), X^c)\} && \text{(Boundary Region)}\\
=& \ \underline{G}(X) \Delta \overline{G}(X),
\end{aligned} \tag{2.5}$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ for $A, B \subseteq U$.

It is noteworthy that the intuition behind Equation 2.5 is very similar to that of [18] (Equation 1). Refer to [9] for more discussion on this point. It is also the case that $\mathrm{POS}_G(X) = \mathrm{NEG}_G(X^c)$ and $\mathrm{BR}_G(X) = \mathrm{BR}_G(X^c)$ for any $X \subseteq W$ [9]. We will routinely use these facts throughout the paper.

Partial Monotonicity. A decider mapping $S : \mathcal{P}(W) \times \mathcal{P}(W) \to \{0, 1\}$ is called *partial monotone* if $X \subseteq Y \subseteq W$ and $S(A, X) = 1$ imply that $S(A, Y) = 1$ for any $A \subseteq W$. An S-approximation space $G = (U, W, T, S)$ with a partial monotone decider mapping S is called a partial monotone S-approximation space. The lower and upper approximation operators and the three decision regions of such S-approximation spaces satisfy several important properties which are listed in the following proposition:

Proposition 2.1 ([10, 12]). *Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. For all $X, Y \subseteq W$, we have:*

(1) $X \subseteq Y$ *implies* $\overline{G}(X) \subseteq \overline{G}(Y)$,
(2) $X \subseteq Y$ *implies* $\underline{G}(X) \subseteq \underline{G}(Y)$,
(3) $\overline{G}(X \cup Y) \supseteq \overline{G}(X) \cup \overline{G}(Y)$,
(4) $\overline{G}(X \cap Y) \subseteq \overline{G}(X) \cap \overline{G}(Y)$,

(5) $\underline{G}(X \cup Y) \supseteq \underline{G}(X) \cup \underline{G}(Y)$,
(6) $\underline{G}(X \cap Y) \subseteq \underline{G}(X) \cap \underline{G}(Y)$,
(7) $\overline{G}(X) = (\underline{G}(X^c))^c$,
(8) $\underline{G}(X) = (\overline{G}(X^c))^c$,
(9) $X \subseteq Y$ *implies* $POS_G(X) \subseteq POS_G(Y)$,
(10) $X \subseteq Y$ *implies* $NEG_G(Y) \subseteq NEG_G(X)$,
(11) $POS_G(X \cup Y) \supseteq POS_G(X) \cup POS_G(Y)$,
(12) $NEG_G(X \cup Y) \subseteq NEG_G(X) \cup NEG_G(Y)$,
(13) $POS_G(X \cap Y) \subseteq POS_G(X) \cap POS_G(Y)$,
(14) $NEG_G(X \cap Y) \supseteq NEG_G(X) \cap NEG_G(Y)$,
(15) $POS_G(X) \cap NEG_G(Y) \subseteq POS_G(X) \cap NEG_G(X \cap Y)$.

Inflection Sets. Partial monotone S-approximation spaces can be represented by an equivalent form, which is called an *inflection set*. A pair $(x, X) \in U \times \mathcal{P}(W)$ is called an *inflection point* with respect to G whenever $S(T(x), X) = 1$ and for all $Y \subsetneq X$, we have $S(T(x), Y) = 0$ [10]. The inflection set of a partial monotone G, which is denoted by $\mathcal{IS}(G)$, is defined as the set of all of its inflection points. Moreover, for $x \in U$ we use $\mathcal{IP}_G(x)$ to represent the collection of $X \subseteq W$ where $(x, X) \in \mathcal{IS}(G)$, so that $\mathcal{IS}(G) = \cup_{x \in U} \{(x, X) | X \in \mathcal{IP}_G(x)\}$.

Trivial Elements. An element $x \in U$ is called *trivial* if we have either $\mathcal{IP}_G(x) = \emptyset$ or $\mathcal{IP}_G(x) = \{\emptyset\}$. In the former case, we have $S(T(x), X) = 0$ for all $X \subseteq W$, so x appears in none of the lower approximations $\underline{G}(X)$ and in every upper approximation $\overline{G}(X^c)$. So, the element x is not providing any useful information, i.e. it cannot be used to distinguish any pair of subsets of W. Similarly, in the latter case, $S(T(x), \emptyset) = 1$, which, due to partial monotonicity, implies $S(T(x), X) = 1$ for all $X \subseteq W$. Hence, for all $X \subseteq W$, we have $x \in \underline{G}(X)$ and $x \notin \overline{G}(X^c)$. So x does not provide any useful information in this case, either.

Reducibility. As argued above, if x is a trivial element, one can remove x and get a smaller system from which one can get just as much information as the initial system. A partial monotone S-approximation space is called *reducible* if it contains a trivial element, otherwise we call it *irreducible*.

3. s-Approximation Spaces and Belief Structures

In this section, we study the relationship between S-approximation spaces and belief structures.

The qualities of lower and upper approximations with respect to an S-approximation space are defined as follows:

Definition 3.1. Let $G = (U, W, T, S)$ be an S-approximation space. The qualities of lower and upper approximations of a set $X \subseteq W$ with respect to G

are defined as:

$$\underline{Q}_G(X) = \frac{|\mathrm{POS}_G(X)|}{|U|}, \tag{3.1}$$

and

$$\overline{Q}_G(X) = \frac{|\mathrm{POS}_G(X)| + |\mathrm{BR}_G(X)|}{|U|}. \tag{3.2}$$

The qualities defined in Equations 3.1 and 3.2 are dual. This is stated more formally in the following proposition:

Proposition 3.2. *Let $G = (U, W, T, S)$ be an S-approximation space. Then, for all $X \subseteq W$ we have $\underline{Q}_G(X) = 1 - \overline{Q}_G(X^c)$.*

Proof. The proof is as follows and uses the fact that $\mathrm{POS}_G(X) = \mathrm{NEG}_G(X^c)$:

$$\begin{aligned}\underline{Q}_G(X) =& \frac{|\mathrm{POS}_G(X)|}{|U|} = \frac{|\mathrm{NEG}_G(X^c)|}{|U|} \\ =& \frac{|U \setminus (\mathrm{POS}_G(X^c) \cup \mathrm{BR}_G(X^c))|}{|U|} \\ =& 1 - \frac{|\mathrm{POS}_G(X^c)| + |\mathrm{BR}_G(X^c)|}{|U|} \\ =& 1 - \overline{Q}_G(X^c).\end{aligned} \tag{3.3}$$

□

Next, we consider the properties of these quality values for a partial monotone S-approximation space.

Proposition 3.3. *Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. Then, $\underline{Q}_G(\emptyset) = 0$.*

Proof. It suffices to show that $\mathrm{POS}_G(\emptyset) = \emptyset$. The proof is by contradiction. Suppose there exists some $x \in U$ such that $x \in \mathrm{POS}_G(\emptyset)$. So, it is the case that $S(T(x), \emptyset) = 1$ and $S(T(x), W) = 0$. This is a contradiction with partial monotonicity of G, since $\emptyset \subseteq W$ and we need to have $S(T(x), W) = 1$. Therefore the desired result is obtained. □

Proposition 3.4. *Let $G = (U, W, T, S)$ be an irreducible partial monotone S-approximation space. Then, $\underline{Q}_G(W) = 1$.*

Proof. Note that G is irreducible, hence for every $x \in U$, there exists $X \subseteq W$, such that $S(T(x), X) = 1$[§]. Therefore, by partial monotonicity, we have $S(T(x), W) = 1$ for all $x \in U$. Moreover, $S(T(x), \emptyset) = 0$ for all $x \in U$. Hence, $x \in \mathrm{POS}_G(W)$ for all $x \in U$ and $\mathrm{POS}_G(W) = U$. So, $\underline{Q}_G(W) = \frac{|U|}{|U|} = 1$. □

[§] Otherwise x is trivial and G is reducible, which is a contradiction.

Proposition 3.5. *Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. Then, for all $\ell \in \mathbb{N}$ we have*

$$\underline{Q}_G(\cup_{i=1}^{\ell} X_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,\ell\}} (-1)^{|I|+1} \underline{Q}_G(\cap_{i \in I} X_i), \tag{3.4}$$

where $X_i \subseteq W$.

Proof. By the definition, we have

$$\underline{Q}_G(\cup_{i=1}^{\ell} X_i) = \frac{\left|\mathrm{POS}_G(\cup_{i=1}^{\ell} X_i)\right|}{|U|}. \tag{3.5}$$

By partial monotonicity of G, we have

$$\frac{\left|\mathrm{POS}_G(\cup_{i=1}^{\ell} X_i)\right|}{|U|} \geq \frac{\left|\cup_{i=1}^{\ell} \mathrm{POS}_G(X_i)\right|}{|U|}, \tag{3.6}$$

since $\mathrm{POS}_G(\cup_{i=1}^{\ell} X_i) \supseteq \cup_{i=1}^{\ell} \mathrm{POS}_G(X_i)$. Now the desired result can be obtained by appling the inclusion-exclusion principle. □

Propositions 3.3 to 3.5 result in the following:

Proposition 3.6. *Let $G = (U, W, T, S)$ be an irreducible partial monotone S-approximation space. The quality of lower approximation, as defined in Definition 3.1, is a belief function.*

Similarly, for an irreducible partial monotone S-approximation space, the quality of upper approximation is a plausibility function. This is treated more formally in the following proposition:

Proposition 3.7. *Let $G = (U, W, T, S)$ be an irreducible partial monotone S-approximation space. Then the quality of upper approximation, as defined in Definition 3.1, is a plausibility function.*

Proof. By the duality of belief and plausibility functions, we have $\mathrm{Pl}_G(X) = 1 - \mathrm{Bel}_G(X^c)$ and this is all we have to show. By the definition, we have

$$\begin{aligned}
\underline{Q}_G(X^c) =& \frac{|\mathrm{POS}_G(X^c)|}{|U|} \\
=& \frac{|\mathrm{NEG}_G(X)|}{|U|} \\
=& \frac{|U \setminus (\mathrm{POS}_G(X) \cup \mathrm{BR}_G(X))|}{|U|} \\
=& 1 - \frac{|\mathrm{POS}_G(X)| + |\mathrm{BR}_G(X)|}{|U|} \\
=& 1 - \overline{Q}_G(X).
\end{aligned} \tag{3.7}$$

By applying Proposition 3.6, the desired result is obtained. □

By Propositions 3.6 and 3.7, it can be said that every irreducible partial monotone S-approximation space induces a belief structure on W.

Theorem 3.8. *Let $G = (U, W, T, S)$ be an irreducible partial monotone S-approximation space. Then, G induces a belief structure $(\mathcal{M}, m)$ on W where*

$$m(X) = \sum_{Y \subseteq X} (-1)^{|X \setminus Y|} \underline{Q}_G(Y), \tag{3.8}$$

and

$$\mathcal{M} = \{X \subseteq W \mid m(X) \neq 0\}, \tag{3.9}$$

for $X \subseteq W$.

Proof. The bpa can be defined from a belief function, which is the quality of lower approximation (by Proposition 3.6), by the following relation

$$n(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \operatorname{Bel}(B), \tag{3.10}$$

where n is a bpa [19]. This concludes the proof. □

Next, we show that belief structures can induce S-approximation spaces.

Theorem 3.9. *Suppose that $(\mathcal{M}, m)$ is a given belief structure over a finite non-empty set W such that for all focal elements $X \in \mathcal{M}$, there exist $a, b \in \mathbb{Z}^+$ such that $m(X) = \frac{a}{b}$. Then, there exists an S-approximation space $G = (U, W, T, S)$ such that the quality of lower and upper approximations with respect to G are the corresponding belief and plausibility functions.*

Proof. The proof is by construction. Without loss of generality, we assume that there exists a constant $d \in \mathbb{Z}^+$ such that for all focal elements $X \in \mathcal{M}$, we have $m(X) = \frac{c}{d}$ for some $c \in \mathbb{Z}^+$. This is easy to obtain by computing the least common multiple.

Now define the set U as $U = \{1, \dots, d\}$. For each $X \in \mathcal{M}$ with $m(X) = \frac{l_X}{d}$, we choose a subset A_X of size l_X of U. We assume that the A_X's are pairwise disjoint. We can always find such disjoint A_X's, since $\sum_{X \in \mathcal{M}} m(X) = 1$ and hence $\sum_{X \in \mathcal{M}} l_X = d$. Now for each $i \in A_X$, we let $T(i) = X$. Finally, we let the decider mapping S be the ordinary set inclusion operator $S_{\subseteq}$.

Next, it is easy to see that G satisfies the conditions of Propositions 3.6 and 3.7. Therefore, the qualities of lower and upper approximations with respect to G are belief and plausibility functions, respectively.

Finally, we show that for all $X \subseteq W$, the belief and plausibility values of X with respect to $(\mathcal{M}, m)$ are equal to the corresponding values with respect to G. Since the belief and plausibility functions are dual, it suffices to show the result for belief. This can be done as follows:

$$\begin{aligned}\underline{Q}_G(X) &= \frac{|\mathrm{POS}_G(X)|}{|U|} \\ &= \frac{|\{x \in U \mid T(x) \subseteq X\}|}{|U|} \\ &= \sum_{Y \subseteq X} m(Y) = \mathrm{Bel}(X).\end{aligned} \tag{3.11}$$

This concludes the proof. □

Now suppose that we are given a belief structure $(\mathcal{M}, m)$ over U and an irreducible partial monotone S-approximation space $G = (U, W, T, S)$. Then we can induce a belief structure $(\mathcal{M}', m')$ on W by declaring $\mathcal{M}'$ as

$$\mathcal{M}' = \{Z \subseteq W \mid \exists x \in U, (x, Z) \in \mathcal{IS}(G)\}, \tag{3.12}$$

and the bpa m' as

$$m'(Y) = \begin{cases} \sum_{X \in \mathcal{M}} \frac{m(X)}{|X|} \times \left(\sum_{x \in X,\ Y \in \mathcal{IP}_G(x)} \frac{1}{|\mathcal{IP}_G(x)|} \right) & \text{if} \quad Y \in \mathcal{M}', \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

The intuition behind Equation 3.13 is that the bpa value of every $X \in \mathcal{M}$ is divided between each $x \in X$ equally likely, which are called their shares. Then, the bpa m' of $Y \in \mathcal{M}'$ receives the shares of those $x \in X$ for which $Y \in \mathcal{IP}_G(X)$.

Theorem 3.10. *Given a belief structure $(\mathcal{M}, m)$ on a finite non-empty set U and an irreducible partial monotone S-approximation space $G = (U, W, T, S)$, $(\mathcal{M}', m')$ as defined in Equations 3.12 and 3.13 is a valid belief structure on U.*

Proof. The bpa m' needs to satisfy two conditions, i.e. (1) $m'(\emptyset) = 0$ and (2) $\sum_{Y \subseteq W} m'(Y) = 1$. By the hypothesis that $\emptyset \notin \mathcal{IP}_G(x)$ for all $x \in U$, we have $\emptyset \notin \mathcal{M}'$ and therefore, its bpa value $m'(\emptyset)$ is zero. The second property can be proven as follows (note that for all $x \in X \in \mathcal{M}$, we have $\mathcal{IP}_G(x) \subseteq \mathcal{M}'$):

$$
\begin{aligned}
\sum_{Y\subseteq W} m'(Y) &= \sum_{Y\in\mathcal{M}'} m'(Y) \\
&= \sum_{Y\in\mathcal{M}'}\sum_{X\in\mathcal{M}} \frac{m(X)}{|X|} \times \left(\sum_{x\in X \ | \ Y\in\mathcal{IP}_G(x)} \frac{1}{|\mathcal{IP}_G(x)|} \right) \\
&= \sum_{x\in X\in\mathcal{M}, Y\in\mathcal{IP}_G(x)\subseteq\mathcal{M}'} \frac{m(X)}{|X|} \times \frac{1}{|\mathcal{IP}_G(x)|} \\
&= \sum_{X\in\mathcal{M}} \frac{m(X)}{|X|} \times \left(\sum_{x\in X, \ Y\in\mathcal{IP}_G(x)} \frac{1}{|\mathcal{IP}_G(x)|} \right) \\
&= \sum_{X\in\mathcal{M}} \frac{m(X)}{|X|} \times \left(\sum_{x\in X} 1 \right) \\
&= \sum_{X\in\mathcal{M}} m(X) = 1.
\end{aligned}
\tag{3.14}
$$

□

4. Conclusion and Future Research Directions

In this paper, we studied some connections between the Dempster-Shafer's theory of evidence and the concept of S-approximation spaces. First, we defined two numeric measures called the qualities of lower and upper approximations for S-approximation spaces. Then, we showed that they can be used to derive a belief structure from an irreducible partial monotone S-approximation space in a natural way. Finally, we showed that given a belief structure on a set U and an irreducible partial monotone S-approximation space $G = (U, W, T, S)$, a valid natural belief structure can be induced on W.

The results obtained in this paper are the first ones settling a relation between the two theories and are extensible by trying to answer the following proposed problems:

(1) Can belief structures be generalized to two universal sets with respect to an arbitrary S-approximation space in a natural or meaningful way?
(2) Can the results of this paper be extended to neighborhood systems, especially the ones in [9]? For example, by fusing knowledge mappings of multiple S-approximation spaces with a similar approach to [5].
(3) Can the qualities of lower and upper approximations be used to reduce the knowledge mappings in the context of [9]? For example, can one find a minimal set of knowledge mappings of multiple S-approximation spaces for which the amount of information one can obtain from that set does not change compared to the case when she uses all of them?

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