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Approximate Solution of Linear Volterra-Fredholm Integral Equations and Systems of Volterra-Fredholm Integral Equations using Taylor Expansion Method

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Abstract. In this study, a new application of Taylor expansion is considered to estimate the solution of Volterra-Fredholm integral equations (VFIEs) and systems of Volterra-Fredholm integral equations (SVFIEs). Our proposed method is based upon utilizing the nth-order Taylor polynomial of unknown function at an arbitrary point and employing integration method to convert VFIEs into a system of linear equations with respect to unknown function and its derivatives. An approximate solution can be easily determined by solving the obtained system. Furthermore, this method leads always to the exact solution if the exact solution is a polynomial function of degree up to n. Also, an error analysis is given. In addition, some problems are provided to demonstrate the validity and applicability of the proposed method.

Keywords: Volterra-Fredholm integral equations, Systems of Volterra-Fredholm integral equations, Mixed Volterra-Fredholm integral equations, Error analysis, Taylor expansion.

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1. Introduction

The Volterra-Fredholm integral equations obtained from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal extension of an epizootic, and moreover from different biological and physical models [1]. Furthermore, differential equations with transformed argument or differential equations of neutral type can be transformed into Volterra-Fredholm integral equations[24]. The Volterra-Fredholm integral equations are considered in the literature in the following two forms

$$\psi(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)\psi(t)dt + \lambda_2 \int_a^b K_2(x,t)\psi(t)dt, \tag{1.1}$$

and the mixed form

$$\psi(x) = f(x) + \lambda \int_a^x \int_a^b K(s,t)\psi(t)dtds, \tag{1.2}$$

where functions $f(x)$, $K_1(x,t)$, $K_2(x,t)$ and $K(s,t)$ are analytic functions and λ_1, λ_2 and λ are arbitrary constants.

It is well-known that the analytical solution of VFIEs generally does not exist except for special cases, and thus, numerical methods to obtain an approximate solution have become the preferred approach for solving VFIEs. Several numerical and approximate methods such as the Adomian decomposition method [1], the modified decomposition method [2], Taylor series method [3], direct and iterative methods [4], pseudospectral methods [5] and other methods [6-10] have been used for solving linear Volterra-Fredholm integral equations. In recent years, many different methods have been proposed to estimate the solution of nonlinear Volterra-Fredholm integral equations, such as collocation method [11], Taylor polynomial method [12], homotopy perturbation method [13], triangular functions methods [14], rationalized Haar functions methods [15, 16], modification of hat functions [17], wavelet method [18] and other methods [19-25].

The method presented in this paper is applicable to a wide range of problems and we summarize the history of some of the problems that have been solved by this method. Xian-Fang Li [26] proposed a novel application of Taylor expansion method for approximate solution of linear ordinary differential equations with variable coefficients. Next, Li and his co-authors expanded the above-mentioned method for solving Abel integral equation [27, 28], Riccati equation [29], an integral equation with fixed singularity for a cruciform crack [30], a class of linear integro-differential equations [31], and fractional integro-differential equations [32]. Vahidi and Didgar improved the Taylor expansion method proposed in [29] for determining the solution of Riccati equations [33]. Didgar and Ahmadi expanded the method proposed in [26] for solving systems of linear ordinary and fractional differential equations [34]. Moreover, Maleknejad and Damercheli [35] developed the method for solving linear second

kind Volterra integral equations system. This work presents a new and simple method based on Taylor expansion [26-35] to solve linear Volterra-Fredholm integral equations and systems of including them. By expanding unknown function to be determined as an nth-order Taylor polynomial and employing integration method, we can convert the VFIE into a system of linear equations with respect to unknown function and its derivatives. An approximate desired solution is determined by solving the resulting system according to a standard rule. In a similar manner, by means of the Taylor expansion for unknown functions, the SVFIEs will be converted into a system of linear equations with respect to unknown functions and their derivatives. As before, a desired solution can be obtained by solving the resulting new system. The results of the obtained numerical approximations of this method are then compared with the referenced methods for different examples. In the present investigation, the main powerful advantage of this method is that an nth-order approximation is equal to exact solution if the exact solution is a polynomial function of at most n.

The rest of this paper is organized as follows. In Section 2, we introduce our method for solving VFIEs and SVFIEs. In Section 3, we give an error analysis. In Section 4, we investigate several numerical examples, which demonstrate the effectiveness of our technique. In Section 5, some tentative conclusions will be drawn.

2. Description of the Method

2.1. **Linear Volterra-Fredholm integral equation of the second kind.** Consider the Volterra-Fredholm integral equation (1.1) as follows

$$\psi(x) + \lambda_1 \int_a^x K_1(x,t)\psi(t)dt + \lambda_2 \int_a^b K_2(x,t)\psi(t)dt = f(x). \tag{2.1}$$

To solve Volterra-Fredholm integral equation (2.1) approximately by following the method used in previous studies [26-35], we convert the VFIE into a system of linear equations with respect to unknown function and its derivatives. Toward this goal, we consider the assumptions below.

Assumption 1: The demanded solution $\psi(t)$ is $n+1$ times continuously differentiable. Therefore, $\psi(t)$ can be expressed in terms of the nth-order Taylor series at an arbitrary point $x \in I$ as

$$\psi(t) = \psi(x) + \psi'(x)(t-x) + \cdots + \frac{1}{n!}\psi^{(n)}(x)(t-x)^n + E_n(t,x), \tag{2.2}$$

in which $E_n(t,x)$ indicates the Lagrange error bound

$$E_n(t,x) = \frac{\psi^{(n+1)}(\xi)}{(n+1)!}(t-x)^{n+1}, \tag{2.3}$$

for some point ξ between x and t.

Assumption 2: Generally, the Lagrange error bound $E_n(t,x)$ becomes sufficiently small as n gets great enough. Especially, if the demanded solution $\psi(t)$ is a polynomial function of degree up to n, then the last Lagrange error bound becomes zero. In other words, the obtained approximate solution of Eq. (2.1) is equal to the demanded true solution.

By omitting the last Lagrange error bound as in Assumption 2, we approximately expand $\psi(t)$ as

$$\psi(t) \approx \sum_{k=0}^{n} \psi^{(k)}(x) \frac{(t-x)^k}{k!}. \tag{2.4}$$

Inserting the approximate relation (2.4), for $\psi(t)$, into Eq. (2.1) leads to

$$\psi(x) + \lambda_1 \sum_{k=0}^{n} \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_a^x K_1(x,t)(x-t)^k dt +$$

$$\lambda_2 \sum_{k=0}^{n} \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_a^b K_2(x,t)(x-t)^k dt = f(x), \tag{2.5}$$

that can be simplified as

$$c_{00}(x)\psi(x) + c_{01}(x)\psi'(x) + \ldots + c_{0n}(x)\psi^{(n)}(x) = f(x), \tag{2.6}$$

where

$$c_{0k}(x) = \delta_{0k} + \frac{(-1)^k}{k!} \left[\lambda_1 \int_a^x K_1(x,t)(x-t)^k dt + \lambda_2 \int_a^b K_2(x,t)(x-t)^k dt \right],$$

$$k = 0, \ldots, n. \tag{2.7}$$

In fact Eq. (2.6) is a linear ordinary differential equation with respect to $\psi(x)$ and its derivatives up to order n. In the following, we want to determine $\psi(x), \ldots, \psi^{(n)}(x)$ by solving a system of linear equations. In order to achieve this goal, other n independent linear equations with respect to $\psi(x), \ldots, \psi^{(n)}(x)$ are needed, which can be achieved by integrating both sides of Eq. (2.1) n times with respect to x from a to s and with the help of changing the order of the integrations. Therefore, we have

$$\int_a^x (x-t)^{i-1}\psi(t)dt + \lambda_1 \int_a^x \int_t^x (x-s)^{i-1} K_1(s,t)\psi(t)dsdt +$$

$$\lambda_2 \int_a^b \int_a^x (x-s)^{i-1} K_2(s,t)\psi(t)dsdt = f_{(i)}(x), \quad i = 1, \ldots, n, \tag{2.8}$$

where

$$f_{(i)}(x) = \int_a^x (x-t)^{i-1} f(t)dt, \quad i = 1, \ldots, n, \tag{2.9}$$

in which the variable s has replaced by x, for simplicity. Similarly, we apply the Taylor expansion again and after substituting (2.4) for $\psi(t)$ into Eq. (2.8), we obtain

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!}\psi^{(k)}(x)\int_a^x(x-t)^{k+i-1}dt+$$

$$\lambda_1\sum_{k=0}^{n}\frac{(-1)^k}{k!}\psi^{(k)}(x)\int_a^x\int_t^x(x-s)^{i-1}(x-t)^kK_1(s,t)dsdt+$$

$$\lambda_2\sum_{k=0}^{n}\frac{(-1)^k}{k!}\psi^{(k)}(x)\int_a^b\int_a^x(x-s)^{i-1}(x-t)^kK_2(s,t)dsdt=f_{(i)}(x),$$

$$i=1,\ldots,n, \tag{2.10}$$

or equivalently

$$c_{i0}(x)\psi(x)+c_{i1}(x)\psi'(x)+\ldots+c_{in}(x)\psi^{(n)}(x)=f_{(i)}(x),\quad i=1,\ldots,n, \tag{2.11}$$

where

$$c_{ik}(x)=\frac{(-1)^k}{k!}(-\frac{(x-a)^{k+i}}{k+i}+\lambda_1\int_a^x\int_t^x(x-s)^{i-1}(x-t)^kK_1(s,t)dsdt+$$

$$\lambda_2\int_a^b\int_a^x(x-s)^{i-1}(x-t)^kK_2(s,t)dsdt),\qquad k=0,\ldots,n. \tag{2.12}$$

In this way, Eqs. (2.6) and (2.11) construct a system of linear equations with respect to the unknown function $\psi(x)$ and its derivatives up to order n. Briefly, this system can be rewritten as follows

$$C(x)\Psi(x)=F(x), \tag{2.13}$$

where

$$C(x)=\begin{bmatrix} c_{00}(x) & c_{01}(x) & \ldots & c_{0n}(x)\\ c_{10}(x) & c_{11}(x) & \ldots & c_{1n}(x)\\ \vdots & \vdots & \ddots & \vdots\\ c_{n0}(x) & c_{n1}(x) & \ldots & c_{nn}(x)\end{bmatrix}, \tag{2.14}$$

$$F(x)=\begin{bmatrix} f(x)\\ f_{(1)}(x)\\ \vdots\\ f_{(n)}(x)\end{bmatrix},\quad \Psi(x)=\begin{bmatrix} \psi(x)\\ \psi'(x)\\ \psi''(x)\\ \vdots\\ \psi^{(n)}(x)\end{bmatrix}. \tag{2.15}$$

In the sequel, making use of a standard rule to the resulting system yields an nth-order approximate solution of Eq. (2.1) as $\psi_n(x)$. We note that not only

$\psi(x)$ but also $\psi^{(i)}(x)$, for $i = 1, \ldots, n$, are determined by solving the resulting system but in point of fact, it is $\psi(x)$ that we need.

2.2. **Mixed Volterra-Fredholm integral equation.** Consider the mixed Volterra-Fredholm integral equation (1.2) as follows

$$\psi(x) + \lambda \int_a^b \int_a^x K(s,t)\psi(t)dsdt = f(x). \tag{2.16}$$

Proceeding as before, to solve the mixed Volterra-Fredholm integral equation (2.16) approximately, we reduce the mixed VFIE into a system of linear equations in unknown function and its derivatives. To achieve this end, substituting the approximate relation (2.4), for unknown function $\psi(t)$, into Eq. (2.16), we obtain

$$\psi(x) + \lambda \sum_{k=0}^{n} \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_a^b \int_a^x K(s,t)(x-t)^k dsdt = f(x). \tag{2.17}$$

Thus Eq. (2.16) is converted into linear ordinary differential equation (2.17) with respect to $\psi(x)$ and its derivatives up to order n. Now, we want to determine $\psi(x), \ldots, \psi^{(n)}(x)$ by solving a system of linear equations. The approach to be used here is identical to the approach used in the previous subsection. This means that we should integrate both sides of the (2.16) n times. Consequently, we have

$$\int_a^x (x-t)^{i-1}\psi(t)dt + \lambda \int_a^b \int_a^x (x-s)^{i-1} K(s,t)\psi(t)dsdt = f_{(i)}(x), \tag{2.18}$$

where

$$f_{(i)}(x) = \int_a^x (x-t)^{i-1} f(t)dt, \quad i = 1, \ldots, n. \tag{2.19}$$

We again apply the Taylor polynomial and after substituting (2.4) for $\psi(t)$ into Eq. (2.18), we obtain

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_a^x (x-t)^{k+i-1} dt +$$

$$\lambda \sum_{k=0}^{n} \frac{(-1)^k}{k!} \psi^{(k)}(x) \int_a^b \int_a^x (x-s)^{i-1} K(s,t)(x-t)^k dsdt = f_{(i)}(x), \tag{2.20}$$

Therefore, Eqs. (2.17) and (2.20) construct a system of linear equations for the unknown function $\psi(x)$ and its derivatives up to order n. We indicate this system as follows

$$C(x)\Psi(x) = F(x), \tag{2.21}$$

where

$$C(x) = \begin{bmatrix} c_{00}(x) & c_{01}(x) & \dots & c_{0n}(x) \\ c_{10}(x) & c_{11}(x) & \dots & c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n0}(x) & c_{n1}(x) & \dots & c_{nn}(x) \end{bmatrix}, \tag{2.22}$$

$$F(x) = \begin{bmatrix} f(x) \\ f_{(1)}(x) \\ \vdots \\ f_{(n)}(x) \end{bmatrix}, \quad \Psi(x) = \begin{bmatrix} \psi(x) \\ \psi'(x) \\ \psi''(x) \\ \vdots \\ \psi^{(n)}(x) \end{bmatrix}. \tag{2.23}$$

In coefficients matrix (2.22), the first row refers to the coefficients of $\psi^{(i)}(x)$ in Eq. (2.17) for $i = 0, \dots, n$ and the other rows refer to the coefficients of $\psi^{(i)}(x)$ in Eq. (2.20) for $i = 0, \dots, n$. Application of a standard method to the obtained system (2.21) results in an approximate solution of Eq. (2.16).

2.3. **System of Volterra-Fredholm integral equation.** A system of linear Volterra-Fredholm integral equations can be considered as follows

$$\psi_i(x) + \lambda_1 \int_a^b \sum_{j=1}^{n} K_{1_{ij}}(x,t)\psi_j(t)dt + \lambda_2 \int_a^x \sum_{j=1}^{n} K_{2_{ij}}(x,t)\psi_j(t)dt = f_i(x),$$

$$i = 1, \dots, n, \tag{2.24}$$

where $f_i(x)$, $K_{1_{ij}}(x,t)$ and $K_{2_{ij}}(x,t)$ are known functions and $\psi_j(x)$ are the unknown functions for $i, j = 1, \dots, n$, with $f_i(x), K_{1_{ij}}(x,t), K_{2_{ij}}(x,t) \in C(I)$, where I is the interval of interest.

In this section we convert the linear Volterra-Fredholm integral equations system (2.24) into a system of linear equations in unknown functions and their derivatives by the following procedure:

The method assumes that the desired solutions $\psi_j(t)$ are $m+1$ times continuously differentiable on the interval I, i.e., $\psi_j \in C^{m+1}(I)$. Therefore, for $\psi_j \in C^{m+1}(I)$, $\psi_j(t)$ can be expressed in terms of the mth-order truncated Taylor expansion at an arbitrary point $x \in I$ as

$$\psi_j(t) \approx \sum_{k=0}^{m} \psi_j^{(k)}(x) \frac{(t-x)^k}{k!}. \tag{2.25}$$

Inserting the approximate relation (2.25), for unknown functions $\psi_j(t)$, into Eq. (2.24), we obtain

$$\psi_i(x) + \lambda_1 \sum_{j=1}^{n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \psi_j^{(k)}(x) \int_a^b K_{1_{ij}}(x,t)(x-t)^k dt$$

$$+\lambda_2 \sum_{j=1}^{n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \psi_j^{(k)}(x) \int_a^x K_{2_{ij}}(x,t)(x-t)^k dt$$

$$= f_i(x), \qquad i = 1, \ldots, n. \tag{2.26}$$

Now, we consider Eq. (2.26) as a linear system of ordinary differential equations with rspect to $\psi_j(x)$ and its derivatives up to order m. In other words, we obtained n linear equations in (2.26) for $n \times (m+1)$ unknowns $\psi_j^{(k)}$, for $k = 0, \ldots, m, j = 1, \ldots, n$. In the following, we want to determine $\psi_j(x), \ldots, \psi_j^{(m)}(x)$ by solving a system of linear equations. In order to achieve this goal, other $n \times m$ independent linear equations with respect to $\psi_j(x), \ldots, \psi_j^{(m)}(x)$ are needed, which can be achieved by integrating both sides of Eq. (2.24) m times with respect to x from a to s and changing the order of the integrations. Thus, we have

$$\int_a^x (x-t)^{l-1} \psi_i(t) dt + \lambda_1 \sum_{j=1}^{n} \int_a^b \int_a^x (x-s)^{l-1} K_{1_{ij}}(s,t) \psi_j(t) ds dt$$

$$+\lambda_2 \sum_{j=1}^{n} \int_a^x \int_t^x (x-s)^{l-1} K_{2_{ij}}(s,t) \psi_j(t) ds dt$$

$$= f_i^{(l)}(x), \qquad l = 1, \ldots, m, \tag{2.27}$$

where

$$f_i^{(l)}(x) = \int_a^x (x-t)^{l-1} f_i(t) dt, \qquad i = 1, \ldots, n, \tag{2.28}$$

in which the variable s has replaced by x, for simplicity. Substituting (2.25) for $\psi_j(t)$ into Eq. (2.27) we have

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \psi_i^{(k)}(x) \int_a^x (x-t)^{l+k-1} dt$$

$$+\lambda_1 \sum_{j=1}^{n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \psi_j^{(k)}(x) \int_a^b \int_a^x (x-s)^{l-1} (x-t)^k K_{1_{ij}}(s,t) ds dt$$

$$+\lambda_2 \sum_{j=1}^{n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \psi_j^{(k)}(x) \int_a^x \int_t^x (x-s)^{l-1} (x-t)^k K_{2_{ij}}(s,t) ds dt =$$

$$f_i^{(l)}(x), \tag{2.29}$$

for $l = 1, \ldots, m$.

Therefore, Eqs. (2.26) and (2.29) construct a system of linear equations for the unknown functions $\psi_j(x)$ and its derivatives up to order m. We rewrite this system as the following compact form

$$C(x)\Psi(x) = F(x), \tag{2.30}$$

where

$$C(x) = \begin{bmatrix} c_{10}^{10}(x) & \cdots & c_{n0}^{10}(x) & \cdots & c_{1k}^{10}(x) & \cdots & c_{nk}^{10}(x) & \cdots & c_{1m}^{10}(x) & \cdots & c_{nm}^{10}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{10}^{n0}(x) & \cdots & c_{n0}^{n0}(x) & \cdots & c_{1k}^{n0}(x) & \cdots & c_{nk}^{n0}(x) & \cdots & c_{1m}^{n0}(x) & \cdots & c_{nm}^{n0}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{10}^{1l}(x) & \cdots & c_{n0}^{1l}(x) & \cdots & c_{1k}^{1l}(x) & \cdots & c_{nk}^{1l}(x) & \cdots & c_{1m}^{1l}(x) & \cdots & c_{nm}^{1l}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{10}^{nl}(x) & \cdots & c_{n0}^{nl}(x) & \cdots & c_{1k}^{nl}(x) & \cdots & c_{nk}^{nl}(x) & \cdots & c_{1m}^{nl}(x) & \cdots & c_{nm}^{nl}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{10}^{1m}(x) & \cdots & c_{n0}^{1m}(x) & \cdots & c_{1k}^{1m}(x) & \cdots & c_{nk}^{1m}(x) & \cdots & c_{1m}^{1m}(x) & \cdots & c_{nm}^{1m}(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{10}^{nm}(x) & \cdots & c_{n0}^{nm}(x) & \cdots & c_{1k}^{nm}(x) & \cdots & c_{nk}^{nm}(x) & \cdots & c_{1m}^{nm}(x) & \cdots & c_{nm}^{nm}(x) \end{bmatrix}, \tag{2.31}$$

$$F(x) = \begin{bmatrix} f_1(x), \ldots, f_n(x), \ldots, f_1^{(l)}(x), \ldots, f_n^{(l)}(x), \ldots, f_1^{(m)}(x), \ldots, f_n^{(m)}(x) \end{bmatrix}^T, \tag{2.32}$$

$$\Psi(x) = \begin{bmatrix} \psi_1(x), \ldots, \psi_n(x), \ldots, \psi_1^{(k)}(x), \ldots, \psi_n^{(k)}(x), \ldots, \psi_1^{(m)}(x), \ldots, \psi_n^{(m)}(x) \end{bmatrix}^T, \tag{2.33}$$

where in coefficients matrix (2.31), the first n rows refer to coefficients of $\psi_j^{(k)}(x)$ in Eq. (2.26) for $k = 0, \ldots, m$, $j = 1, \ldots, n$ and the other rows refer to coefficients of $\psi_j^{(k)}(x)$ in Eq. (2.29) for $l = 1, \ldots, m$. Application of a standard rule to the resulting new system yields an mth-order approximate solution of Eq. (2.24) as $\psi_{j,m}(x)$. We note that not only $\psi_j(x)$ but also $\psi_j^{(k)}(x)$, for $k = 1, \ldots, m$ is determined by solving the resulting new system but in point of fact, it is $\psi_j(x)$ that we need.

3. Error Analysis

In this section, the stability analysis of the scheme is carried out and we expand the error analysis proposed in [28] for derived mth-order approximate solution of Volterra-Fredholm integral equations system (2.24) in order to get theoretical features about the convergence of the suggested method. Suppose that the exact solutions are infinitely differentiable in the interval I; so $\psi_j(t)$ can be expanded as a uniformly convergent Taylor series in I as follows

$$\psi_j(t) = \sum_{k=0}^{\infty} \psi_j^{(k)}(x) \frac{(t-x)^k}{k!}. \tag{3.1}$$

Using the proposed method in the previous section, SVFIEs can be converted into an equivalent system of linear equations with respect to unknown functions $\psi_i^{(k)}(x)$, $k = 0, 1, \ldots$ as

$$\mathbf{C\Psi} = \mathbf{F}, \tag{3.2}$$

where

$$\mathbf{C} = \lim_{n\longrightarrow\infty} \mathbf{C}_{nn}^{nn}, \qquad \mathbf{\Psi} = \lim_{n\longrightarrow\infty} \mathbf{\Psi}_n, \qquad \mathbf{F} = \lim_{n\longrightarrow\infty} \mathbf{F}_n, \tag{3.3}$$

in which $\mathbf{C}_{nn}^{nn}$, $\mathbf{\Psi}_n$ and $\mathbf{F}_n$, as shown in the previous section, are defined as follows

$$\mathbf{C}_{nn}^{nn} = \left[c_{ij}^{pq}(x)\right]_{n(m+1)\times n(m+1)} \quad , \quad \mathbf{\Psi}_n = \left[\psi_i^{(k)}(x)\right]_{n(m+1)\times 1} \quad ,$$

$$\mathbf{F}_n = \left[f_i^{(l)}(x)\right]_{n(m+1)\times 1}. \tag{3.4}$$

Hence, under the solvability conditions for the above system and letting $\mathbf{B} = \mathbf{C}^{-1}$, the unique solution of system (3.2) is represented as

$$\mathbf{\Psi} = \mathbf{BF}. \tag{3.5}$$

We rewrite relation (3.5) in an alternative matrix form as

$$\begin{bmatrix} \mathbf{\Psi}_n \\ \mathbf{\Psi}_\infty \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{nn}^{nn} & \mathbf{B}_{n\infty}^{n\infty} \\ \mathbf{B}_{\infty n}^{\infty n} & \mathbf{B}_{\infty\infty}^{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{F}_n \\ \mathbf{F}_\infty \end{bmatrix}. \tag{3.6}$$

Accordingly, we can find out that the vector $\mathbf{\Psi}_n$ consists of the first $n(m+1)$ elements of the exact solution vector $\mathbf{\Psi}$ must satisfy the following relation

$$\mathbf{\Psi}_n = \mathbf{B}_{nn}^{nn}\mathbf{F}_n + \mathbf{B}_{n\infty}^{n\infty}\mathbf{F}_\infty \tag{3.7}$$

According to the proposed process in this paper, the unique solution of SVFIEs (1.1) can be denoted as

$$\widetilde{\mathbf{\Psi}}_n = \mathbf{C}_{nn}^{nn^{-1}}\mathbf{F}_n, \tag{3.8}$$

where $\mathbf{\Psi}_n$ is replaced by $\widetilde{\mathbf{\Psi}}_n$ as its approximate solution.

Subtracting (3.8) from (3.7) leads to

$$\mathbf{\Psi}_n - \widetilde{\mathbf{\Psi}}_n = \mathbf{D}_{nn}^{nn}\mathbf{F}_n + \mathbf{B}_{n\infty}^{n\infty}\mathbf{F}_\infty \tag{3.9}$$

where $\mathbf{D}_{nn}^{nn} = \mathbf{B}_{nn}^{nn} - \mathbf{C}_{nn}^{nn^{-1}}$.

Now, we expand the right-hand side of (3.9) and the first n elements of the vector at the left-hand side of (3.9) can be expressed as

$$\psi^n(x) - \widetilde{\psi}^n(x) = \sum_{j=0}^{m}\sum_{i=1}^{n} d_{ij}^{p0}(x) f_i^{(j)}(x) + \sum_{j=m+1}^{\infty}\sum_{i=1}^{n} b_{ij}^{p0}(x) f_i^{(j)}(x),$$

$$p = 1, \ldots, n, \tag{3.10}$$

where

$$\psi^n(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_n(x) \end{bmatrix}, \qquad \widetilde{\psi}^n(x) = \begin{bmatrix} \widetilde{\psi}_1(x) \\ \widetilde{\psi}_2(x) \\ \vdots \\ \widetilde{\psi}_n(x) \end{bmatrix}, \tag{3.11}$$

and $d_{ij}^{p0}(x)$, $b_{ij}^{p0}(x)$ are the elements of $\mathbf{D}_{nn}^{nn}$ and $\mathbf{B}_{n\infty}^{n\infty}$, respectively. Thus, according to the Cauchy-Schwarz inequality we have

$$\left|\psi^n(x)-\widetilde{\psi}^n(x)\right| \leq \left(\sum_{j=0}^{m}\sum_{i=1}^{n}\left|d_{ij}^{p0}(x)\right|^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{m}\sum_{i=1}^{n}\left|f_i^{(j)}(x)\right|^2\right)^{\frac{1}{2}}+$$

$$\left(\sum_{j=m+1}^{\infty}\sum_{i=1}^{n}\left|b_{ij}^{p0}(x)\right|^2\right)^{\frac{1}{2}}\left(\sum_{j=m+1}^{\infty}\sum_{i=1}^{n}\left|f_i^{(j)}(x)\right|^2\right)^{\frac{1}{2}}. \tag{3.12}$$

It is to be noted that since $\lim_{n\longrightarrow\infty}\mathbf{D}_{nn}^{nn}=0$ and $\lim_{n\longrightarrow\infty}\mathbf{B}_{n\infty}^{n\infty}=0$, we can obtain $\lim_{n\longrightarrow\infty}|\psi^n(x)-\widetilde{\psi}^n(x)|=0$.

Remark: We remark that the reader can easily understand that the error analysis of Eq. (1.1) or Eq. (1.2) is a special case of error analysis of Volterra-Fredholm integral equations system (2.24) proposed in Section 3.

4. Numerical Examples

In this section, we present numerical results for some Volterra-Fredholm integral equations to illustrate the efficiency and the accuracy of the proposed method. Comparing this method with other methods confirms the validity and applicability of the presented method. All computations are performed using Mathematica 8.

Example 4.1. [4] In Eq. (1.1), take $a=0, b=1, \lambda_1=\lambda_2=\frac{1}{15}$ and $K_1(x,t)=K_2(x,t)=e^{x+t}$. When the exact solution is, $\psi(x)=1$,
$f(x)=\frac{1}{15}\left(15-e^x(e^x+e-2)\right)$. When the exact solution is, $\psi(x)=x$, $f(x)=\frac{1}{15}\left(15x-e^x(xe^x-e^x+2)\right)$. When the exact solution is, $\psi(x)=e^x$, $f(x)=\frac{1}{30}\left(e^x(32-e^2-e^{2x})\right)$.

We use the method described in this paper to determine the approximate solutions. For the cases where $\psi(x)=1$ and $\psi(x)=x$, we can find that $\psi_n(x)$ yields the exact solution only by setting $n=1$. This can be explained as a result of the fact that if a desired solution is a polynomial function of degree n, the nth-order approximate solution leads to its exact solution. For the case where $\psi(x)=e^x$, a comparison between the exact and approximate solutions at ten equidistant points in $[0,1]$ is made by setting $n=1,\ldots,5$ in Table 1. The obtained results show that the exact solution of Eq. (1.1) will be obtained if the true solution is a polynomial function and when the exact solution is not a polynomial, several lower-order approximations will result in an approximate solution with high accuracy.

It is important to note that after converting equations of example 4.1 into a system of linear equations the Mathematica command 'LinearSolve' is used for the obtained system.

TABLE 1. Absolute errors of exmaple 4.1 for $\psi(x) = e^x$.

x	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.1	3.76349×10^{-2}	9.15552×10^{-3}	1.75082×10^{-3}	3.09725×10^{-4}	1.32425×10^{-3}
0.2	3.55673×10^{-2}	8.21867×10^{-3}	1.48846×10^{-3}	2.23263×10^{-4}	9.75866×10^{-6}
0.3	3.26865×10^{-2}	7.15838×10^{-3}	1.22253×10^{-3}	1.72639×10^{-4}	2.22512×10^{-5}
0.4	2.88896×10^{-2}	5.99291×10^{-3}	9.60332×10^{-4}	1.26978×10^{-4}	1.43368×10^{-5}
0.5	2.40705×10^{-2}	4.75186×10^{-3}	7.10331×10^{-4}	8.73098×10^{-5}	9.15104×10^{-6}
0.6	1.81219×10^{-2}	3.48005×10^{-3}	4.82069×10^{-4}	5.45819×10^{-5}	5.25494×10^{-6}
0.7	1.09379×10^{-2}	2.24327×10^{-3}	2.86014×10^{-4}	2.94739×10^{-5}	2.57127×10^{-6}
0.8	2.41911×10^{-3}	1.13668×10^{-3}	1.33242×10^{-4}	1.22867×10^{-5}	9.51633×10^{-7}
0.9	7.51321×10^{-3}	2.97965×10^{-4}	3.48824×10^{-5}	2.79036×10^{-6}	1.85437×10^{-7}
1.0	1.88859×10^{-2}	7.16497×10^{-5}	1.11951×10^{-6}	1.44462×10^{-8}	7.31082×10^{-11}

This example was used in [4] and has been solved by fixed point and collocation methods. We present the results obtained in [4] in Tables 2, 3 and 4. $R1$ and $R2$ denote the maximum error in the fixed point and in the collocation methods, respectively.

TABLE 2. Example 1 for $\psi(x) = 1$ (Method in [4]).

	$n = 17, m = 7$	$n = 33, m = 8$
R1	3.1×10^{-5}	8.5×10^{-6}
R2	5.7×10^{-6}	1.1×10^{-6}

TABLE 3. example 4.1 for $\psi(x) = x$ (Method in [4]).

	$n = 17, m = 1$	$n = 33, m = 6$
R1	1.1×10^{-16}	1.1×10^{-16}
R2	9.9×10^{-16}	5.9×10^{-16}

TABLE 4. example 4.1 for $\psi(x) = e^x$ (Method in [4]).

	n=17, m=7	n=33, m=8
$R1$	1.6×10^{-4}	1.4×10^{-5}
$R2$	3.1×10^{-5}	6.3×10^{-6}

EXAMPLE 4.2. [4] In Eq. (1.1), take $a = 0, b = 1, \lambda_1 = \lambda_2 = \frac{1}{3}$ and $K_1(x,t) = K_2(x,t) = \sin x \cos t$. When the exact solution is, $\psi(x) = x$, $f(x) = \frac{1}{3}(3x - \sin x(x \sin x + \cos x + \sin 1 + \cos 1 - 2))$. When the exact solution is, $\psi(x) = x^2$, $f(x) = \frac{1}{3}(3x^2 - \sin x(x^2 \sin x + 2x \cos x - 2 \sin x - \sin 1 + 2 \cos 1))$. When the exact solution is, $\psi(x) = e^x$, $f(x) = \frac{1}{3}(3e^x - \sin x(e^x(\sin x + \cos x) + e(\sin 1 + cos1) - 2)/2)$.

We apply the method described in this paper to obtain the approximate solutions. For the case where $\psi(x) = x$, the first-order approximate solution and for the case where $\psi(x) = x^2$, the second-order approximate solution yield the exact solution as expected, since the nth-order approximate solution yields the exact solution if the exact solution is a polynomial of degree up to n. For the case where $\psi(x) = x^2$ the absolute errors between the exact solution and its approximations are tabulated in Table 5 by setting $n = 1, 2$. For the case where $\psi(x) = e^x$, a comparison between the exact and approximate solutions at ten equidistant points in $[0, 1]$ is made by setting $n = 1, \ldots, 4$ in Table 6. The obtained results and Tables 5 and 6 show that the exact solution of Eq. (1.1) will be obtained if the true solution is a polynomial and when the exact solution is not a polynomial, several lower-order approximations will result in an approximate solution with high accuracy.

TABLE 5. Absolute errors of example 4.2 for $\psi(x) = x^2$.

x	n=1	n=2
0.1	7.29623×10^{-3}	0
0.2	1.13452×10^{-2}	0
0.3	1.27889×10^{-2}	0
0.4	1.22808×10^{-2}	0
0.5	1.05010×10^{-2}	0
0.6	8.17093×10^{-3}	0
0.7	6.06066×10^{-3}	0
0.8	4.98645×10^{-3}	0
0.9	5.79465×10^{-3}	0
1.0	9.33236×10^{-3}	0

This example was used in [4] and has been solved by fixed point and collocation methods. We present the results obtained in [4] in Tables 7, 8 and 9. $R1$ and $R2$ denote the maximum error in the fixed point and in the collocation methods, respectively.

TABLE 6. Absolute errors of example 4.2 for $\psi(x) = e^x$.

x	n=1	n=2	n=3	n=4
0.1	4.95836×10^{-3}	1.08386×10^{-3}	1.93742×10^{-4}	2.73441×10^{-5}
0.2	8.09986×10^{-3}	1.69868×10^{-3}	2.89073×10^{-4}	4.14711×10^{-5}
0.3	9.59723×10^{-3}	1.92722×10^{-3}	3.11122×10^{-4}	4.21918×10^{-5}
0.4	9.68830×10^{-3}	1.85608×10^{-3}	2.83057×10^{-4}	3.61018×10^{-5}
0.5	8.69991×10^{-3}	1.57522×10^{-3}	2.25687×10^{-4}	2.68959×10^{-5}
0.6	7.07687×10^{-3}	1.17681×10^{-3}	1.57021×10^{-4}	1.73385×10^{-5}
0.7	5.41246×10^{-3}	7.53099×10^{-4}	9.17188×10^{-5}	9.28091×10^{-6}
0.8	4.47603×10^{-3}	3.92898×10^{-4}	4.04468×10^{-5}	3.70062×10^{-6}
0.9	5.23246×10^{-3}	1.76566×10^{-4}	9.20418×10^{-6}	7.73258×10^{-7}
1.0	8.84927×10^{-3}	1.69535×10^{-4}	1.25787×10^{-6}	3.21934×10^{-9}

TABLE 7. example 4.2 for $\psi(x) = x$ (Method in [4]).

	n=17, m=2	n=33, m=1
$R1$	1.1×10^{-16}	2.2×10^{-16}
$R2$	2.1×10^{-15}	1.5×10^{-15}

TABLE 8. example 4.2 for $\psi(x) = x^2$ (Method in [4]).

	n=17, m=8	n=33, m=8
$R1$	3.8×10^{-4}	9.5×10^{-5}
$R2$	7.8×10^{-5}	4.7×10^{-5}

TABLE 9. example 4.2 for $\psi(x) = e^x$ (Method in [4]).

	n=17, m=7	n=33, m=8
$R1$	2.1×10^{-4}	1.3×10^{-5}
$R2$	4.1×10^{-5}	1.1×10^{-5}

EXAMPLE 4.3. Consider the following VFIE [5]

$$\psi(x) = f(x) + \int_0^x e^{x-t}\psi(t)dt - \int_0^1 e^{x+t}\psi(t)dt, \tag{4.1}$$

where

$$f(x) = x^2 + (e-4)e^x + x^2 + 2x + 2. \tag{4.2}$$

The exact solution of this equation is given by $\psi(x) = x^2$. The VFIE (4.1) was solved using our proposed method in this paper. A comparison between the exact and approximate solutions at ten equidistant points in $[0, 1]$ is made by

setting $n = 1, 2$ in Table 10, and we can find that the second-order approximate solution yields the exact solution, as expected.

TABLE 10. Absolute errors of example 4.3.

x	n=1	n=2
0.1	1.44347×10^{-1}	0
0.2	1.33488×10^{-1}	0
0.3	1.19631×10^{-1}	0
0.4	1.02793×10^{-1}	0
0.5	8.30517×10^{-2}	0
0.6	6.05618×10^{-2}	0
0.7	3.55929×10^{-2}	0
0.8	8.57831×10^{-3}	0
0.9	1.98063×10^{-2}	0
1.0	4.85126×10^{-2}	0

This example was used in [5] and has been solved using Taylor collocation method [23] and a subclass of spectral methods called pseudospectral method. Table 11 shows the approximation errors in L^∞ norm for the Taylor collocation method and pseudospectral method. It is easy to see that Taylor collocation method has a mildly growing error when N is increased.

TABLE 11. L^∞ errors of example 4.3 for the Taylor collocation and pseudospectral methods [5].

N	2	5	8	11	14
Taylor	9.9920×10^{-16}	1.5543×10^{-15}	2.4425×10^{-15}	5.7436×10^{-15}	1.8382×10^{-14}
pseudospectral	3.2125×10^{-1}	7.8639×10^{-5}	9.1700×10^{-9}	2.5507×10^{-13}	1.1241×10^{-15}
N	17	20	23	26	29
Taylor	3.8591×10^{-13}	3.9024×10^{-13}	6.3076×10^{-12}	4.0158×10^{-11}	2.7326×10^{-10}
pseudospectral	8.8818×10^{-16}	4.0176×10^{-15}	1.7764×10^{-15}	4.9821×10^{-15}	2.5882×10^{-15}
N	32	35	38	41	44
Taylor	2.2233×10^{-9}	3.3849×10^{-9}	1.1361×10^{-8}	8.3156×10^{-9}	2.6713×10^{-9}
pseudospectral	2.5535×10^{-15}	5.1001×10^{-15}	2.7756×10^{-15}	1.7764×10^{-15}	6.5781×10^{-15}

EXAMPLE 4.4. Consider the following VFIE [14, 17, 36]

$$\psi(x) = -x^4 - x^3 + 12x^2 - x - 5 + \int_0^x (x - t)\psi(t)dt + \int_0^1 (x + t)\psi(t)dt, \quad (4.3)$$

with the exact solution $\psi(x) = 12x^2 + 6x$. According to the proposed technique, we obtain the approximate results by setting $n = 1, 2$ and we observe that the second-order approximate solution yields the exact solution. This is

easily understood since the exact solution is a polynomial function of degree 2. Moreover, the obtained L^2-norm of error by the present method are compared with the rationalized hat functions method [36], triangular functions method [14] and modification hat functions method [17] in Table 12.

TABLE 12. Comparison of L^2 errors of example4.4.

Methods	Errors
Hat functions method [36]	
$n = 8$	$1.4e-1$
$n = 16$	$3.6e-2$
$n = 32$	$8.9e-3$
Triangular functions method [14]	
$n = 8$	$5.3e-2$
$n = 16$	$1.8e-2$
$n = 32$	$3.7e-3$
Modification hat functions method [17]	
$n = 8$	$4.3e-3$
$n = 16$	$2.9e-4$
$n = 32$	$1.7e-5$
Present method	
$n = 1$	$1.5e-1$
$n = 2$	0

EXAMPLE 4.5. Consider the following mixed Vloterra-Fredholm integral equation [1]

$$\psi(x) = 2 + 4x - \frac{9}{8}x^2 - 5x^3 + \int_0^x \int_0^1 (s-t)\psi(t)dtds, \tag{4.4}$$

with the exact solution $\psi(x) = 2+3x-5x^3$. Applying the proposed technique, we obtain the approximate results by setting $n = 1, 2, 3$. The corresponding absolute errors at equidistant points in $[0, 1]$ are listed in Table 13 and we observe that the third-order approximate solution yields the exact solution, as expected.

TABLE 13. Comparison of the absolute errors of example 4.5.

x	n=1	n=2	n=3
0.1	7.02662×10^{-2}	6.25905×10^{-2}	0
0.2	1.25225×10^{-1}	9.04786×10^{-2}	0
0.3	1.59088×10^{-1}	9.49063×10^{-2}	0
0.4	1.69476×10^{-1}	8.48151×10^{-2}	0
0.5	1.56519×10^{-1}	6.71228×10^{-2}	0
0.6	1.21752×10^{-1}	4.70002×10^{-2}	0
0.7	6.68383×10^{-2}	2.81487×10^{-2}	0
0.8	7.81955×10^{-3}	1.30764×10^{-2}	0
0.9	1.04481×10^{-1}	3.37580×10^{-3}	0
1.0	2.29412×10^{-1}	0	0

Example 4.6. Consider the following system of Volterra-Fredholm integral equations [37]

$$\begin{cases} \psi_1(x) + \int_0^1 \frac{1}{2}\cos(x-t)\psi_1(t)dt + \int_0^x \frac{1}{2}\cos(x-t)\psi_1(t)dt + \\ \int_0^1 \frac{1}{2}\sin(x-t)\psi_2(t)dt + \int_0^x \frac{1}{2}\sin(x-t)\psi_2(t)dt = f_1(x), \\ \psi_2(x) + \int_0^1 \frac{1}{2}\cos(x-t)\psi_1(t)dt + \int_0^x \frac{1}{2}\cos(x-t)\psi_1(t)dt + \\ \int_0^1 \frac{1}{2}\sin(x-t)\psi_2(t)dt + \int_0^x \frac{1}{2}\sin(x-t)\psi_2(t)dt = f_2(x). \end{cases}$$

For the exact solutions $\psi_1(x) = x^2$ and $\psi_2(x) = x^3$, $f_1(x) = 1/2(x^3 + 2x^2 - 4x - 3\cos(x-1) - 2\sin(x-1) + 8\sin x)$ and $f_2(x) = 1/2(3x^3 - 4x - 3\cos(x-1) - 2\sin(x-1) + 8\sin x)$. Using our proposed method, we evaluate the approximate results by setting $m = 1, 2, 3$ and the obtained absolute errors are listed in Tables 14 and 15. We observe that the accuracy is quite satisfactory and the third-order approximate solution yields the exact solution, as expected.

Table 14. Absolute errors of example 4.6 for $\psi_1(x)$.

	$e_{1,m}(x)$		
x	m=1	m=2	m=3
0.1	1.59772×10^{-2}	5.19473×10^{-2}	0
0.2	1.41536×10^{-2}	3.19795×10^{-2}	0
0.3	1.41449×10^{-2}	1.79354×10^{-2}	0
0.4	1.43504×10^{-2}	8.72979×10^{-3}	0
0.5	1.36872×10^{-2}	3.26497×10^{-3}	0
0.6	1.15963×10^{-2}	4.93114×10^{-4}	0
0.7	8.00933×10^{-3}	5.36278×10^{-4}	0
0.8	3.27865×10^{-3}	6.44342×10^{-4}	0
0.9	1.92477×10^{-3}	5.04925×10^{-4}	0
1.0	6.73539×10^{-3}	6.38956×10^{-4}	0

This example was used in [37] and has been solved by the collocation method. We present the maximum of the absolute errors obtained from Ref. [37] in Table 16.

TABLE 15. Absolute errors of example 4.6 for $\psi_2(x)$.

	$e_{2,m}(x)$		
x	m=1	m=2	m=3
0.1	1.59772×10^{-2}	5.19473×10^{-2}	0
0.2	1.41536×10^{-2}	3.19795×10^{-2}	0
0.3	1.41449×10^{-2}	1.79354×10^{-2}	0
0.4	1.43504×10^{-2}	8.72979×10^{-3}	0
0.5	1.36872×10^{-2}	3.26497×10^{-3}	0
0.6	1.15963×10^{-2}	4.93114×10^{-4}	0
0.7	8.00933×10^{-3}	5.36278×10^{-4}	0
0.8	3.27865×10^{-3}	6.44342×10^{-4}	0
0.9	1.92477×10^{-3}	5.04925×10^{-4}	0
1.0	6.73539×10^{-3}	6.38956×10^{-4}	0

TABLE 16. The maximum of the absolute errors in[37].

x^2	1.77×10^{-15}
x^3	2.55×10^{-15}

In the other hand, for the exact solutions $\psi_1(x) = x^2$ and $\psi_2(x) = x^4$, $f_1(x) = 1/2(x^4 - 12x^2 + 2x + 24 + 15\cos(x-1) - 19\sin(x-1) - 4\sin x - 48\cos x)$ and $f_2(x) = 1/2(3x^4 - 12x^2 + 2x + 24 + 15\cos(x-1) - 19\sin(x-1) - 4\sin x - 48\cos x)$. Clearly, according to the present technique the forth-order approximation yields the exact solution. This is easily understood since the exact solution is a polynomial function of degree 4. This case also has been solved by the collocation method in [37] and we present the maximum of the absolute errors obtained from Ref. [37] in Table 17.

TABLE 17. The maximum of the absolute errors in [37].

	$n = 11$	$n = 31$	$n = 101$
x^2	1.46×10^{-6}	2.09×10^{-8}	1.77×10^{-10}
x^4	7.39×10^{-5}	920×10^{-7}	7.48×10^{-9}

5. CONCLUSION

In this paper, an efficient method based on Taylor expansion has been suggested to determine the approximate solution of Volterra-Fredholm integral equations and systems of Volterra-Fredholm integral equations. An interesting advantage of the proposed method is to find the exact solution if the problem has the exact solution as polynomial function. Moreover, an error analysis was established. The method can be easily extended to systems of Volterra-Fredholm integro-differential equations.

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