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On the Smoothness of Functors

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Abstract. In this paper we will try to introduce a good smoothness notion for a functor. We consider properties and conditions from geometry and algebraic geometry which we expect a smooth functor should have.

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1. Introduction

Nowadays noncommutative algebraic geometry is in the focus of many basic topics in mathematics and mathematical physics. In these fields, any under consideration space is an abelian category and a morphism between noncommutative spaces is a functor between abelian categories. So one may ask to generalize some aspects of morphisms between commutative spaces to morphisms between noncommutative ones. One of the important aspects in commutative case is the notion of smoothness of a morphism which is stated in some languages, for example: by lifting property as a universal language, by projectivity of relative cotangent sheaves as an algebraic language and by inducing a surjective morphism on tangent spaces as a geometric language.

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In this paper, in order to generalize the notion of smooth morphism to a functor we propose three different approaches. A glance description for the first one is as follows: linear approximations of a space are important and powerful tools. They have geometric meaning and algebraic structures such as the vector space of the first order deformations of a space. So it is legitimate to consider functors which preserve linear approximations. On the other hand first order deformations are good candidates for linear approximations in categorical settings. These observations make it reasonable to consider functors which preserve first order deformations.
The second one is motivated from both Schlessinger's approach and simultaneous deformations. Briefly speaking, a simultaneous deformation is a deformation which deforms some ingredients of an object simultaneously. Deformations of morphims with nonconstant target, deformations of a couple $(X, \mathcal{L})$, in which X is a scheme and $\mathcal{L}$ is a line bundle on X, are examples of such deformations. Also we see that by this approach one can get a morphism of moduli spaces of some moduli families. We get this, by fixing a universal ring for objects which correspond to each other by a smooth functor. Theorem 3.1 connects this notion to the universal ring of an object. In 3.1 and 3.2 we describe geometrical setting and usage of this approach respectively.
The third notion of smoothness comes from a basic reconstruction theorem of A. Rosenberg, influenced by ideas of A. Grothendieck. We think that this approach can be a source to translate other notions from commutative case to noncommutative one. In remarks 3.2 and 4.1 we notice that these three smoothness notions are independent of each other.

Throughout this paper **Art** will denote the category of Artinian local k-algebras with quotient field k. By **Sets**, we denote the category of sets which its morphisms are maps between sets. Let F and G be functors from **Art** to **Sets**. For two functors $F, G : \mathbf{Art} \to \mathbf{Sets}$ the following is the notion of smoothness between morphisms of F and G which has been introduced in [8]:

A morphism $D : F \to G$ between covariant functors F and G is said to be a smooth morphism of functors if for any surjective morphism $\alpha : B \to A$, with $\alpha \in \mathrm{Mor}(\mathbf{Art})$, the morphism

$$F(B) \to F(A) \underset{G(A)}{\times} G(B)$$

is a surjective map in **Sets**.
Note that this notion of smoothness is a notion for morphisms between special functors, i.e. functors from the category **Art** to the category **Sets**, while the concepts for smoothness which we introduce in this paper are notions for functors, but not for morphisms between them.

A functor F : **Art** $\to$ **Sets** is said to be a deformation functor if it satisfies in definition 2.1. of [5]. For a fixed field k the schemes in this paper are schemes over the scheme $\operatorname{Spec}(k)$ otherwise it will be stated.

2. First Smoothness notion and some examples

Definition 2.1. *Let M and C be two categories. We say that the category C is a multicategory over M if there exists a functor $T : C \to M$, in which for any object A of M, $T^{-1}(A)$ is a full subcategory of C.*
Let C and $\overline{C}$ be two multicategories over M and $\overline{M}$ respectively. A morphism of multicategories C and $\overline{C}$ is a couple (u, ν) of functors, with $u : C \to \overline{C}$ and $\nu : M \to \overline{M}$ such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{f} & M \\ u\downarrow & & \downarrow \nu \\ \overline{C} & \to & \overline{M} \end{array}$$

The category of modules over the category of rings and the category of sheaves of modules over the category of schemes are examples of multicategories.

Definition 2.2. *For a S-scheme X and $A \in$ **Art**, we say that $\mathcal{X}$ is a S-deformation of X over A if there is a commutative diagram:*

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ S & \to & S \underset{k}{\times} A \end{array}$$

in which X is a closed subscheme of $\mathcal{X}$, the scheme $\mathcal{X}$ is flat over $S \underset{k}{\times} A$ and one has $X \cong S \underset{S \underset{k}{\times} A}{\times} \mathcal{X}$.

Note that in the case $S = \operatorname{Spec}(k)$, we would have the usual deformation notion and as in the usual case the set of isomorphism classes of first order S-deformations of X is a k-vector space. The addition of two deformations $(\mathcal{X}_1, \mathcal{O}_{\mathcal{X}_1})$ and $(\mathcal{X}_2, \mathcal{O}_{\mathcal{X}_2})$ is denoted by $(\mathcal{X}_1 \bigcup_X \mathcal{X}_2, \mathcal{O}_{\mathcal{X}_1} \underset{\mathcal{O}_X}{\times} \mathcal{O}_{\mathcal{X}_2})$.

Definition 2.3. **i)** *Let C be a category. We say C is a category with enough deformations, if for any object c of C, one can associate a deformation functor. We will denote the associated deformation functor of c, by D_c. Moreover for any $c \in \operatorname{Obj}(C)$ let $D_c(k[\epsilon])$ be the tangent space of c, where $k[\epsilon]$ is the ring of dual numbers.*
ii) *Let C_1 and C_2 be two multicategories with enough deformations over Sch/k, and (F, id) be a morphism between them. We say F is a smooth functor if it has the following properties:*

1 : *For any object M of C_1, if M_1 is a deformation of M in C_1 then $F(M_1)$ is a deformation of $F(M)$ on A in C_2.*
2 : *The map*

$$\begin{array}{ccc} D_M(k[\varepsilon]) & \to & D_{F(M)}(k[\varepsilon]) \\ \mathcal{X} & \mapsto & F(\mathcal{X}) \end{array}$$

is a morphism of tangent spaces.

The following are examples of categories with enough deformations:
1) Category of schemes over a field k.
2) Category of coherent sheaves on a scheme X.
3) Category of line bundles over a scheme.
4) Category of algebras over a field k.

We will need the following lemma to present an example of smooth functors:

Lemma 2.1. *Let X, X_1, X_2 and $\mathcal{X}$ be schemes over a fixed scheme S. Assume that the following diagram of morphisms between schemes is a commutative diagram.*

$$\begin{array}{ccc} X & \xrightarrow{i_1} & X_1 \\ \downarrow & & \downarrow g \\ X_2 & \xrightarrow[i_2]{} & \mathcal{X} \end{array}$$

If i_1 is homeomorphic on its image, then so is i_2.

Proof. See Lemma (2.5) of [9]. □

Example 2.1. *Let Y be a flat scheme over S. Then the fibered product by Y over S is smooth. More precisely, the functor:*

$$\begin{array}{rcl} F : \mathrm{Sch}/S & \to & \mathrm{Sch}/Y \\ F(X) & = & X \underset{S}{\times} Y \end{array}$$

is smooth.

Let X be a closed subscheme of $\mathcal{X}$. Then $X \underset{S}{\times} Y$ is a closed subscheme of $\mathcal{X} \underset{S}{\times} Y$. To get the flatness of $\mathcal{X} \underset{S}{\times} Y$ over $S \underset{k}{\times} A$, it suffices to have the flatness of Y over S. It can also be verified easily that the isomorphism:

$$(\mathcal{X} \underset{S}{\times} Y) \underset{S \underset{k}{\times} A}{\times} S \cong X \underset{S}{\times} Y$$

is valid. Therefore $\mathcal{X} \underset{S}{\times} Y$ is a S-deformation of $X \underset{S}{\times} Y$ if $\mathcal{X}$ is such a deformation of X. This verifies the first condition of item (**ii**) of definition 2.3. To prove the second condition we need the following:

Lemma 2.2. *Let Y, X_1 and X_2 be S-schemes. Assume that X is a closed subscheme of X_1 and X_2. Then we have the following isomorphism:*

$$(X_1 \bigcup_X X_2) \underset{S}{\times} Y \cong (X_1 \underset{S}{\times} Y) \bigcup_{X \underset{S}{\times} Y} (X_2 \underset{S}{\times} Y).$$

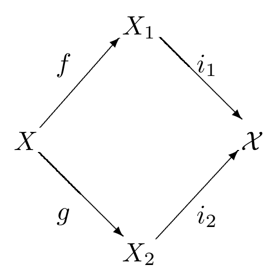
Proof. For simplicity we set:

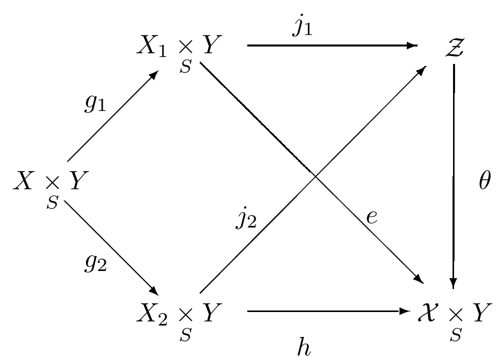
$$X_1 \underset{X}{\cup} X_2 = \mathcal{X} \qquad , \qquad (X_1 \underset{S}{\times} Y) \bigcup_{X \underset{S}{\times} Y} (X_2 \underset{S}{\times} Y) = \mathcal{Z}$$

By universal property of $\mathcal{Z}$ we have a morphism $\theta : \mathcal{Z} \to \mathcal{X} \underset{S}{\times} Y$. We prove that θ is an isomorphism. Let $i_1 : X_1 \to \mathcal{X}$, $i_2 : X_2 \to \mathcal{X}$, $j_1 : X_1 \underset{S}{\times} Y \to \mathcal{Z}$ and $j_2 : X_2 \underset{S}{\times} Y \to \mathcal{Z}$ be the inclusion morphisms. Set theoretically we have:

$$\begin{array}{rcll} j_1(X_1 \underset{S}{\times} Y) \bigcup j_2(X_2 \underset{S}{\times} Y) & = & \mathcal{Z} & \text{(I)} \\ i_1(X_1) \bigcup i_2(X_2) & = & \mathcal{X} & \text{(II)} \end{array}$$

Now consider the following commutative diagrams:

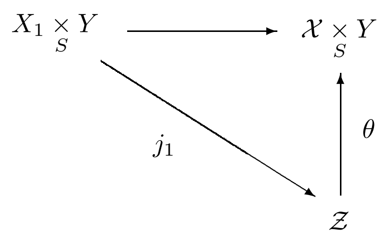




Let $z \in \mathcal{X} \underset{S}{\times} Y$, $\alpha = P_{\mathcal{X}}(z) \in \mathcal{X}$ and $\beta = P_Y(z) \in Y$ in which $P_{\mathcal{X}}$ and P_Y are the first and second projections from $\mathcal{X} \underset{S}{\times} Y$ to $\mathcal{X}$ and Y respectively. Then by relation (II) one has $\alpha \in i_1(X_1)$ or $\alpha \in i_2(X_2)$. If $\alpha = i_1(\alpha_1) \in i_1(X_1)$, then α_1 and β go to the same element in S by η_{X_1} and η_Y in which $\eta_{X_1} : X_1 \to S$ and

$\eta_Y : Y \to S$ are the maps which make X_1 and Y schemes over S. Therefore there exists an element γ in $X_1 \underset{S}{\times} Y$ such that $\overline{P}_{X_1}(\gamma) = \alpha_1$ and $\overline{P}_Y(\gamma) = \beta$ in which $\overline{P}_{X_1}$ and $\overline{P}_Y$ are the first and second projections from $X \underset{S}{\times} Y$ to X_1 and Y respectively. By universal property of fibered products γ belongs to $\mathcal{X} \underset{S}{\times} Y$ and $\theta(\gamma) = z$. The proof for the case $\alpha \in i_2(X)$ is similar. This implies that θ is surjective.

For injectivity of θ assume that $\theta(z_1) = \theta(z_2)$. The relation (I) implies that z_1 and z_2 belong to $\operatorname{im}(j_1) \bigcup \operatorname{im}(j_2)$. Set $z_1 = j_1(c_1)$ and $z_2 = j_2(c_2)$. There are two cases: if $z_1, z_2 \in \operatorname{im}(j_1) \cap \operatorname{im}(j_2)$, then the lemma 2.1 implies $e(c_1) \neq e(c_2)$ when $c_1 \neq c_2$. Now by commutativity of the subdiagram:



we have $\theta(z_1) \neq \theta(z_2)$ when $z_1 \neq z_2$.

Otherwise assume that $z_1 \in \operatorname{im}(j_1)$ and $z_2 \in \operatorname{im}(j_2) - \operatorname{im}(j_1)$. In this case one can see easily that $i_1\overline{P}_{X_1}(c_1) = i_2 q_2(c_2)$ in which q_2 is the first projection from $X_2 \underset{S}{\times} Y$ to X_2. Since $\mathcal{X}$ is the fibered sum of X_1 and X_2, there exists an element $x \in X$ such that $i_1 f(x) = i_2 g(x)$, $f(x) = \overline{P}_{X_1}(c_1)$ and $g(x) = q_2(c_2)$.

Set $y = p_2 e(c_1)$ in which p_2 is the second projection from $\mathcal{X} \underset{S}{\times} Y$ to Y. By a diagram chasing we see that x and y go to the same element in S. This implies that there exists an element ϵ in $X \underset{S}{\times} Y$ which is mapped to x and y by first and second projections, respectively. Also it is easy to see that the equalities $g_1(x, y) = c_1$ and $g_2(x, y) = c_2$ are valid. Since $\mathcal{Z}$ is the fibered sum of $X_1 \underset{S}{\times} Y$ and $X_2 \underset{S}{\times} Y$ on $X \underset{S}{\times} Y$, we have $z_1 = z_2$ which means that θ is injective. This together with the surjectivity of θ implies that θ is bijective. Continuity of θ and its inverse, follow by a diagram chasing.

Finally we should prove that $\mathcal{O}_{\mathcal{X} \underset{S}{\times} Y} \cong \mathcal{O}_Z$. Since the claim is local, it is sufficient to prove it for affine schemes. Let $\mathcal{X}$ be an affine scheme, so X_1, X_2 and X are affine schemes, since they are closed subschemes of $\mathcal{X}$ each one defined by a nilpotent sheaf of ideals. Set $\mathcal{X} = \operatorname{Spec}(A)$, $X_1 = \operatorname{Spec}(A_1)$, $X_2 = \operatorname{Spec}(A_2)$, $X = \operatorname{Spec}(A_0)$, $Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(C)$. The isomorphism $\mathcal{O}_{\mathcal{X} \underset{S}{\times} Y} \cong \mathcal{O}_Z$ reduces to the following isomorphism:

$$(A_1 \underset{A_0}{\times} A_2) \underset{C}{\otimes} B \cong (A_1 \underset{C}{\otimes} B) \underset{A_0 \underset{C}{\otimes} B}{\times} (A_2 \underset{C}{\otimes} B).$$

Define a morphism as follows:

$$d : (A_1 \underset{A_0}{\times} A_2) \underset{C}{\otimes} B \quad \to \quad (A_1 \underset{C}{\otimes} B) \underset{A_0 \underset{C}{\otimes} B}{\times} (A_2 \underset{C}{\otimes} B)$$
$$d((a_1, a_2) \otimes b) = (a_1 \otimes b, a_2 \otimes b).$$

By a simple commutative algebra argument it can be shown that this is in fact an isomorphism. This completes the proof of lemma. □

This lemma shows that the fibered product functor, induces an additive homomorphism on tangent spaces. To check linearity with respect to scalar multiplication, take an element a in the field k. Multiplication by a is a ring homomorphism on D. This homomorphism induces a morphism from $S \underset{k}{\times} D$ to $S \underset{k}{\times} D$ and scalar multiplication on t_{D_X}, comes from composition of this map with π. In other words this gives a map from $\mathcal{X} \underset{S}{\times} Y$ into $\mathcal{X} \underset{S}{\times} Y$. These together give the linearity of homomorphism induced from F with respect to scalar multiplication.
This observation together with the lemma 2.2, give the smoothness of the fibered product functor.

Lemma 2.3. *Let X and Y be arbitrary schemes and assume that there exist morphisms h and g from η to η_1 and η_2, where η, η_1, η_2 are sheaves of $\mathcal{O}_X$-modules on the scheme X. Then for any morphism $f : X \to Y$ we have the following isomorphisms:*

$$f_*(\eta_1 \underset{\eta}{\times} \eta_2) \cong f_*(\eta_1) \underset{f_*(\eta)}{\times} f_*(\eta_2)$$
$$f^*(\rho_1 \underset{\rho}{\times} \rho_2) \cong f^*(\rho_1) \underset{f^*(\rho)}{\times} f^*(\rho_2).$$

Proof. For the first isomorphism, it is enough to consider the definition of direct image of sheaves.
To prove the second one, assume that $(M_i)_{i \in I}, (N_i)_{i \in I}$ and $(P_i)_{i \in I}$ are direct systems of modules over a directed set I. We have to prove that

$$\lim_{i \in I}(M_i \underset{P_i}{\times} N_i) \cong (\lim_{i \in I}(M_i)) \underset{(\lim_{i \in I}(P_i))}{\times} (\lim_{i \in I}(N_i)).$$

The above isomorphism can be proved by elementary calculations and using elementary properties of direct limits. □

Example 2.2. *Let $f : X \to Y$ be a flat morphism of schemes. Then f_* and f^* are smooth functors.*

In fact let η be a coherent sheaf on X and $\eta_1 \in \mathrm{Coh}(X \underset{k}{\times} D)$ be a deformation of η. By these assumptions we would have:

$$(f_*(\eta)) \underset{D}{\otimes} k = f_*(\eta_1 \underset{D}{\otimes} k) = f_*(\eta).$$

Moreover $f_*(\eta_1)$ is flat on D, because η is flat on D. This implies that f_* satisfies in the first condition of smoothness. The second one is the first isomorphism of lemma 2.3. Therefore f_* is smooth. Smoothness of f^* is similar to that of f_*.

Assuming this notion of smoothness we can generalize another aspect of geometry to categories.

Definition 2.4. *Let C be a category with enough deformations. We define the tangent category of C, denoted by TC, as follows:*

$$\begin{aligned} \mathrm{Obj}(TC) &:= \bigcup_{c \in \mathrm{Obj}(C)} T_c C \\ \mathrm{Mor}_{TC}(\upsilon, \omega) &:= \mathrm{Mor}(V, W) \end{aligned}$$

which by T_cC, we mean the tangent space of D_c. Moreover υ and ω are first order deformations of V and W.

Remark 2.1. *(i) It is easy to see that a smooth functor induces a covariant functor on the tangent categories.*
(ii) Let C be an abelian category. Then its tangent category is also abelian.

The following is a well known suggestion of A. Grothendieck: Instead of working with a space, it is enough to work on the category of quasi coherent sheaves on this space. This suggestion was formalized and proved by P. Gabriel for noetherian schemes and in its general form by A. Rosenberg. To do this, Rosenberg associates a locally ringed space to an abelian category A. In a special case he gets the following:

Theorem 2.1. *Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $A = \mathrm{QCoh}(X)$. Then*

$$(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) = (X, \mathcal{O}_X)$$

where $\mathrm{Spec}(A)$ *is the ringed space which is constructed from an abelian category by A. Rosenberg.*

Proof. See Theorem $(A.2)$ of [7]. □

The definition of tangent category and theorem 2.1 motivates the following questions which the authors could not find any positive or negative answer to them until yet.

Question 1: For a fixed scheme X consider $T\,\mathrm{QCoh}(X)$ and TX, the tangent category of category of quasi coherent sheaves on X and the tangent bundle of X respectively. Can TX be recovered from $T\,\mathrm{QCoh}(X)$ by Rosenberg construction?

Question 2: Let $\mathcal{M}$ be a moduli family with moduli space M. Consider $\mathcal{M}$ as a category and consider its tangent category $T\mathcal{M}$. Is there a reconstruction from $T\mathcal{M}$ to TM?

3. Second Smoothness Notion

Definition 3.1. *Let $F : \mathrm{Sch}/k \to \mathrm{Sch}/k$ be a functor with the following property:*
For any scheme X and an algebra $A \in \mathrm{Obj}(\boldsymbol{Art})$, $F(\mathcal{X})$ is a deformation of $F(X)$ over A if $\mathcal{X}$ is a deformation of X over A.
We say F is smooth at X, if the morphism of functors

$$\Theta_X : D_X \to D_{F(X)}$$

is a smooth morphism of functors in the sense of Schlessinger (See [8]). F is said to be smooth if for any object X of Sch/k, the morphism of functors Θ_X is smooth.

The following lemma describes more properties of smooth functors.

Lemma 3.1. (a) *Assume that C_1, C_2 and C_3 are multicategories over the category Sch/k. Let $F_1 : C_1 \to C_2$ and $F_2 : C_2 \to C_3$ be smooth functors with the first notion. Then so is their composition.*
(b) *Let $F_1 : \mathrm{Sch}/k \to \mathrm{Sch}/k$ and $F_2 : \mathrm{Sch}/k \to \mathrm{Sch}/k$ be smooth functors with second notion. Then so is their composition.*
(c) *Let $F : \mathrm{Sch}/k \to \mathrm{Sch}/k$ and $G : \mathrm{Sch}/k \to \mathrm{Sch}/k$ be functors to which F and GoF are smooth with second notion. Then G is a smooth functor.*
(d) *Let $F, G, H : \mathrm{Sch}/k \to \mathrm{Sch}/k$ be smooth functors in the sense of second notion with morphisms of functors $F \to G$ and $H \to G$ between them. Then the functor $F \underset{G}{\times} H$ is smooth functor with the second one.*

Proof. Part (a) of lemma is trivial.
(b) Let $X \in \mathrm{Sch}/k$ and $B \to A$ be a surjective morphism in **Art**. By smoothness of F_1, F_2 and by remark 2.4 of [8], there exists a surjective map

$$\Theta_{F_2(X),F_2oF_1(X)} : D_{F_2oF_1(X)}(B) \underset{D_{F_2oF_1(X)}(A)}{\times} D_X(A) \to D_{F_1(X)}(B) \underset{D_{F_1(X)}(A)}{\times} D_X(A)$$

such that we have

$$\Theta_{X,F_2oF_1(X)} = \Theta_{F_2(X),F_2oF_1(X)} o \Theta_{X,F_2(X)}$$

in which $\Theta_{X,F_2(X)}$ is the surjective map induced by smoothness of F_2. From this equality it follows the map $\Theta_{X,F_2oF_1(X)}$ is surjective immediately.
(c) For a scheme X in the category Sch/k consider a surjective morphism $B \to A$ in **Art**. By smoothness of F, the morphism $D_X \to D_{F(X)}$ is a surjective morphism of functors. Now apply proposition (2.5) of [8] to finish the proof.
(d) Let $X \in \mathrm{Sch}/k$ and $B \to A$ be a surjective morphism in **Art**. Consider the following commutative diagram:

$$\begin{array}{ccc} D_X & \longrightarrow & D_{F(X)} \\ & \searrow & \uparrow \\ & & D_{G(X)} \end{array}$$

Since the morphisms of functors $D_X \to D_{F(X)}$ and $D_X \to D_{G(X)}$ are smooth morphisms of functors, proposition 2.5(iii) of [8] implies that $D_{F(X)} \to D_{G(X)}$ is a smooth morphism of functors. Similarly $D_{H(X)} \to D_{G(X)}$ is a smooth morphism of functors. Again by 2.5(iv) of [8], the morphism of functors:

$$D_{H(X)} \underset{D_{G(X)}}{\times} D_{F(X)} \to D_{H(X)}$$

is a smooth morphism of functors. Since in the diagram:

$$\begin{array}{ccc} D_X & \longrightarrow & D_{H(X)} \underset{D_{G(X)}}{\times} D_{F(X)} \\ & \searrow & \uparrow \\ & & D_{H(X)} \end{array}$$

the morphisms $D_X \to D_{H(X)}$ and $D_{H(X)} \underset{D_{G(X)}}{\times} D_{F(X)}$ are smooth morphisms of functors, part (c) of this lemma implies that $D_{H(X)} \underset{D_{G(X)}}{\times} D_{F(X)}$ is smooth. This completes the proof. □

Remark 3.1. ***(i)*** *The same proof works to generalize part* (c) *of lemma 3.1 as follows:*
($\acute{c}$) *Let* $F : \mathrm{Sch}/k \to \mathrm{Sch}/k$ *and* $G : \mathrm{Sch}/k \to \mathrm{Sch}/k$ *be functors with GoF smooth and F surjective in the level of deformations in the sense that for any* $X \in \mathrm{Sch}/k$ *and any* $A \in \mathrm{Obj}(\mathbf{Art})$ *the morphism* $D_X(A) \to D_{F(X)}(A)$ *is surjective in* **Art**. *Then G is smooth.*
(ii) *One may ask to find a criterion to determine smoothness of a functor. We could not get a complete answer to this question. But by the following fact, one may answer the question at least partially:*
A functor $F : \mathrm{Sch}/k \to \mathrm{Sch}/k$ *is not smooth at X if there exists an algebra* $A \in \mathbf{Art}$ *such that the map* $D_X(A) \to D_{F(X)}(A)$ *is not surjective in* **Art**, *(See* [8]*).*

Theorem 3.1 relates the second smoothness notion to the hull of deformation functors. Recall the hull of a functor is defined in [8]. We need the following:

Lemma 3.2. *Let F : **Art** $\to$ Sets be a functor. Then its hulls are non-canonically isomorphic if there exist.*

Proof. See Proposition 2.9 of [8]. □

Theorem 3.1. *Let $F : \mathrm{Sch}/k \to \mathrm{Sch}/k$ be a functor and for a scheme X the functor F has the following properties:*
(a) $F(\mathcal{X})$ is a deformation of $F(X)$ if $\mathcal{X}$ is a deformation of X.
(b) The functor F induces isomorphism on tangent spaces.
Then F is smooth at X if and only if $(R, F(\xi))$ is a hull of $D_{F(X)}$ whenever (R, ξ) is a hull of D_X.

Proof. To prove the Theorem it is enough to apply parts $(b), (c)$ of lemma 3.1, and lemma 3.2 to the functors

$$\Theta_X : D_X \to D_{F(X)} \quad , \quad h_{R,X} : h_R \to D_X \quad , \quad h_{R,F(X)} : h_R \to D_{F(X)}.$$

□

For a scheme X let:

$$\{\text{pairs } (\mathcal{X}, \Omega_{\mathcal{X}/k}) \text{ which } \mathcal{X} \text{ is an infinitesimal deformation of } X \text{ over } A\ \}$$

be the isomorphism classes of fibered deformations of X.
In the following example we use this notion of deformations of schemes.

Example 3.1. *The functor defined by:*

$$\begin{array}{rcl} F : \mathrm{Sch}/k & \to & \mathrm{QCoh} \\ F(X) & = & \Omega_{X/k} \end{array}$$

is a smooth functor.

Note that if one considers deformations of $\Omega_{X/k}$ as usual case, the above functor will not be smooth. The usual deformation of $\Omega_{X/k}$ can be described as simultaneous deformation of an object, and differential forms on that object. Also this observation is valid for TX and ω_X instead of Ω_X.

Remark 3.2. *The first and second smoothness notions are in general different. Note that a functor which is smooth with the second notion induces surjective maps on tangent spaces. Since the morphism induced on tangent spaces with first notion of smoothness is not necessarily surjective, a functor which is smooth in the sense of first notion is not necessarily smooth with the sense of second notion. Also a functor which is smooth in the sense of second notion can not be necessarily smooth with the first notion in general. In fact the map induced on tangent spaces by second notion is not necessarily a linear map. It is easy to see that the example 3.1 is smooth with both of the notions, but examples 2.1 and 2.2 are smooth just in the sense of first one.*

3.1. **A Geometric interpretation.** Let F be a smooth functor at X. By theorem 3.1, X and $F(X)$ have the same universal rings and this can be interpreted as we are deforming X and $F(X)$ simultaneously. Therefore we have an algebraic language for simultaneous deformations. The example 3.1 can be interpreted as follows: we are deforming a geometric space and an ingredient of that space, e.g. the structure sheaf of the space or its sheaf of relative differential forms, and these operations are smooth.

3.2. **Relation with smoothness of a morphism.** Let $\mathcal{M}$ be a moduli family of algebro - geometric objects with a variety M as its fine moduli space and suppose $Y(m) \to M$ is the fiber on $m \in M$. With this assumptions we would have the following bijections:

$$\begin{aligned} T_{m,M} &\cong \quad \mathrm{Hom}(\mathrm{Spec}(k[\epsilon]), M) \\ &\cong \{\text{classes of first order deformations of X over A}\} \end{aligned}$$

In fact these bijections states that why deformations are important in geometric usages. Now suppose we have two moduli families $\mathcal{M}_1$ and $\mathcal{M}_2$ with varieties M_1 and M_2 as their fine moduli spaces. Also describe $\mathcal{M}_1$ and $\mathcal{M}_2$ as categories in which there exists a smooth functor F between them. In this setting, if we have a morphism between them, induced from F, then it is a smooth morphism.

4. Third Smoothness Notion

This notion of smoothness is completely motivated from Rosenberg's reconstruction theorem, Theorem $(A.2)$ of [7]. For this notion of smoothness we do not use deformation theory.

Definition 4.1. *Let $F : C_1 \to C_2$ be a functor between abelian categories such that there exists a morphism*

$$f : \mathrm{Spec}(C_1) \to \mathrm{Spec}(C_2)$$

induced by the functor F. We say F is a smooth functor if f is a smooth morphism of schemes.

Remark 4.1. (a) *Since this smoothness notion uses a language completely different from the two previous ones, it does not imply non of them and vice versa. We did not verified this claim with details but it is not so legitimate to expect that this smoothness implies the previous ones, because deformation theory is not consistent with the Rosenberg construction. This observation together with the remark 3.2 show that these three notions are independent of each other, having nice geometric and algebraic meaning in their own rights separately.*
(b) *It seems that a functor of abelian categories induces a morphism of schemes in rarely cases. But the cases in which this happens are the cases of enough importance to consider them. Here we mention some cases which this happens.*

(i) *Let* $f : X \to \mathrm{Spec}(k)$ *be a morphism of finite type between schemes. Then it can be shown that* f *is induced by*

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Spec}(k))$$

by Rosenberg's construction. This example is important because it can be a source of motivation, to translate notions from commutative case to noncommutative one.
(ii) *Also the following result of Rosenberg is worth to note:*

Proposition 4.1. *Let* A *be an abelian category.*
(a) For any topologizing subcategory T *of* A*, the inclusion functor* $T \to A$ *induces an embedding* $\mathrm{Spec}(T) \to \mathrm{Spec}(A)$*.*
(b) For any exact localization $Q : A \to A/S$ *and for any* $P \in \mathrm{Spec}(A)$*, either* $P \in \mathrm{Obj}(S)$ *or* $Q(P) \in \mathrm{Spec}(A/S)$*; hence* Q *induces an injective map from* $\mathrm{Spec}(A) - \mathrm{Spec}(S)$ *to* $\mathrm{Spec}(A/S)$*.*

Proof. See Proposition $(A.0.3)$ of [7]. □

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