On generalization of Čebyšev type inequalities

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ABSTRACT. In this paper, we establish new Čebyšev type integral inequalities involving functions whose derivatives belong to L_p spaces via certain integral identities.

1. INTRODUCTION

Let $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous functions whose derivatives $f', g' \in L_\infty[a, b]$. The Čebyšev functional is defined by:

$$(1.1) \quad T(f,g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right)$$

and the following inequality (see [3]) holds:

$$(1.2) \qquad |T(f,g)| \le \frac{1}{12}(b-a)^2 \left\| f' \right\|_\infty \left\| g' \right\|_\infty.$$

Many researchers have given considerable attention to (1.2) and a number of extensions, generalizations and variants have appeared in the literature, see ([1],[2],[4] [5] and [6]) and the references given therein.

Pachpatte in [5] established new inequalities of the Čebyšev type:

Theorem 1. *Let* $f, g : [a, b] \to \mathbb{R}$ *be absolutely continuous functions on* $[a, b]$ *with* $f', g' \in L_2[a, b]$, *then*

$$|P(F, G, f, g)| \leq \frac{(b-a)^2}{12} \left[\frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \left[\frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}$$

and

$$|P(A, B, f, g)| \leq \frac{(b-a)^2}{12} \left[\frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \left[\frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}$$

where

$$(1.3) \quad P(\alpha, \beta, f, g) = \alpha\beta - \frac{1}{b-a} \left(\alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \right) + \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right)$$

$$[f; a, b] = \frac{f(b) - f(a)}{b - a}, \; F = \frac{f(a) + f(b)}{2}, \; G = \frac{g(a) + g(b)}{2}, \; A = f(\frac{a+b}{2}), \; B = g(\frac{a+b}{2})$$

and

$$\|f\|_2 = \left(\int_a^b f^2(x)dx \right)^{\frac{1}{2}}.$$

In [6] Pachpatte presented an additional Čebyšev type inequality in the following theorem:

Theorem 2. *Let* $f, g : [a, b] \to \mathbb{R}$ *be absolutely continuous functions whose derivatives* $f', g' \in L_p[a, b]$, $p > 1$, *then we have*

$$|P(C, D, f, g)| \leq \frac{1}{(b-a)^2} \left(\frac{(2^{q+1} + 1)(b-a)^{q+1}}{3(q+1)6^q} \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p$$

where $P(\alpha, \beta, f, g)$ *is as defined in (1.3)*,

$$C = \frac{1}{3} \left(\frac{f(a) + f(b)}{2} + 2f(\frac{a+b}{2}) \right), \quad D = \frac{1}{3} \left(\frac{g(a) + g(b)}{2} + 2g(\frac{a+b}{2}) \right)$$

with $\frac{1}{p} + \frac{1}{q} = 1$, *and*

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The main purpose of the present note is to establish inequalities similar to the above inequalities.

2. Main Results

Theorem 3. *Let* $f, g : [a, b] \to \mathbb{R}$ *be absolutely continuous functions whose derivatives* $f', g' \in L_p[a, b]$, $p > 1$, *then we have*

$$|P(F, G, f, g)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p$$

where $P(\alpha, \beta, f, g)$ *is as defined in (1.3),*

$$F = \frac{f(a) + f(b)}{2}, \ G = \frac{g(a) + g(b)}{2}$$

with $\frac{1}{p} + \frac{1}{q} = 1$, *and*

$$\|f\|_p = \left(\int\limits_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Proof. From the hypothesis of Theorem 3, we have the following identities (see, [1]),

$$(2.1) \qquad F - \frac{1}{b-a} \int_a^b f(t)dt = L(f; a, b)$$

$$(2.2) \qquad G - \frac{1}{b-a} \int_a^b g(t)dt = L(g; a, b)$$

where

$$L(f; a, b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(t - s) \, dt ds.$$

Multiplying the left sides and right sides of (2.1) and (2.2), we write

$$(2.3) \qquad P(F, G, f, g) = L(f; a, b) L(g; a, b)$$

From (2.3), we get

$$(2.4) \qquad |P(F, G, f, g)| \leq |L(f; a, b)| \, |L(g; a, b)|.$$

Using the Hölder's integral inequality, we obtain
$$(2.5)$$
$$|L(f; a, b)| \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f'(t) - f'(s)| \, |t - s| \, dt ds$$

$$\leq \frac{1}{2(b-a)^2} \left(\int_a^b \int_a^b |f'(t) - f'(s)|^p \, dt ds \right)^{1/p} \left(\int_a^b \int_a^b |t - s|^q \, dt ds \right)^{1/q}.$$

By simple computation,

$$\int_a^b \int_a^b |t-s|^q \, dtds = \int_a^b \left\{ \int_a^s (s-t)^q \, dt + \int_s^b (t-s)^q \, dt \right\} ds$$

$$= \frac{1}{q+1} \int_a^b \left\{ -(s-t)^{q+1} \Big|_a^s + (t-s)^{q+1} \Big|_s^b \right\} ds$$

(2.6)
$$= \frac{1}{q+1} \int_a^b \left\{ (s-a)^{q+1} + (b-s)^{q+1} \right\} ds$$

$$= \frac{1}{(q+1)(q+2)} \left((s-a)^{q+2} - (b-s)^{q+2} \right) \Big|_a^b$$

$$= \frac{2(b-a)^{q+2}}{(q+1)(q+2)}$$

and using $(a+b)^r \leq 2^{r-1}(a^r + b^r)$, $r \geq 1$, $a > 0$, $b > 0$, we have

(2.7)
$$\int_a^b \int_a^b |f'(t) - f'(s)|^p \, dtds \leq \int_a^b \int_a^b \left\{ (|f'(t)| + |f'(s)|)^p \right\} dtds$$

$$\leq 2^{p-1} \int_a^b \int_a^b \left(|f'(t)|^p + |f'(s)|^p \right) dtds$$

$$= 2^{p-1} \left\{ \int_a^b \int_a^b |f'(t)|^p \, dtds + \int_a^b \int_a^b |f'(s)|^p \, dtds \right\}$$

$$= 2^p (b-a) \|f'\|_p^p.$$

Using (2.6) and (2.7) in (2.5), we obtain

(2.8)
$$|L(f;a,b)| \leq \frac{1}{2(b-a)^2} \left(\frac{2(b-a)^{q+2}}{(q+1)(q+2)} \right)^{1/q} (2^p (b-a))^{1/p} \|f'\|_p$$

$$= \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{1/q} \|f'\|_p.$$

Similarly, we have

(2.9)
$$|L(g;a,b)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{1/q} \|g'\|_p.$$

Thus, using (2.8) and (2.9) in (2.4), we obtain

$$|P(F,G,f,g)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{2/q} \|f'\|_p \|g'\|_p.$$

$\square$

Theorem 4. *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$, then we have*

$$(2.10) \qquad |P(A, B, f, g)| \leq \left(\frac{2(b - a)}{(q + 1)(q + 2)} \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p$$

where $P(\alpha, \beta, f, g)$ is as defined in (1.3),

$$A = f(\frac{a + b}{2}), \ B = g(\frac{a + b}{2})$$

with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Proof. From the hypothesis of Theorem 4, we have the following identities (see, [2])

$$(2.11) \qquad A - \frac{1}{b - a} \int_a^b f(t) dt = M(f; a, b)$$

$$(2.12) \qquad B - \frac{1}{b - a} \int_a^b g(t) dt = M(g; a, b)$$

where

$$M(f; a, b) = \frac{1}{2(b - a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(m(t) - m(s)) \, dt ds,$$

and $m(t)$ involved in the notation $M(; a, b)$ is given by

$$m(t) = \begin{cases} t - a, & \text{if } t \in \left(a, \frac{a+b}{2}\right] \\ t - b, & \text{if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Multiplying the left sides and right sides of (2.11) and (2.12), we obtain

$$P(A, B, f, g) = M(f; a, b) M(g; a, b)$$

and thus

$$(2.13) \qquad |P(A, B, f, g)| = |M(f; a, b)| |M(g; a, b)|.$$

From Hölder's integral inequality, we have

$$(2.14) \qquad |M(f; a, b)| \leq \frac{1}{2(b-a)^2} \left(\int_a^b \int_a^b |f'(t) - f'(s)|^p \, dt ds \right)^{1/p} \left(\int_a^b \int_a^b |m(t) - m(s)|^q \, dt ds \right)^{1/q}.$$

On the other hand, we get,

$$\int_a^b \int_a^b |m(t) - m(s)|^q \, dt ds$$

$$= \int_a^b \left\{ \int_a^{(a+b)/2} |t - a - m(s)|^q \, dt + \int_{(a+b)/2}^b |t - b - m(s)|^q \, dt \right\} ds$$

(2.15)
$$= \int_a^{(a+b)/2} \int_a^{(a+b)/2} |t - s|^q \, dt ds + \int_a^{(a+b)/2} \int_{(a+b)/2}^b |t - s + a - b|^q \, dt ds$$

$$+ \int_{(a+b)/2}^b \int_a^{(a+b)/2} |t - s + b - a|^q \, dt ds + \int_{(a+b)/2}^b \int_{(a+b)/2}^b |t - s|^q \, dt ds$$

$$= I_1 + I_2 + I_3 + I_4.$$

Here, by simple computation, we deduce:

$$I_1 = \int_a^{(a+b)/2} \int_a^{(a+b)/2} |t - s|^q \, dt ds$$

$$= \int_a^{(a+b)/2} \left\{ \int_a^s (s - t)^q \, dt + \int_s^{(a+b)/2} (t - s)^q \, dt \right\} ds$$

$$= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ -(s - t)^{q+1} \Big|_a^s + (t - s)^{q+1} \Big|_s^{(a+b)/2} \right\} ds$$

(2.16)
$$= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ (s - a)^{q+1} - \left(\frac{a+b}{2} - s \right)^{q+1} \right\} ds$$

$$= \frac{1}{(q+1)(q+2)} \left\{ (s - a)^{q+2} - \left(\frac{a+b}{2} - s \right)^{q+2} \right\} \Big|_a^{(a+b)/2}$$

$$= \frac{2 \left(\frac{b-a}{2} \right)^{q+2}}{(q+1)(q+2)},$$

$$I_2 = \int_a^{(a+b)/2} \int_{(a+b)/2}^b |t - s + a - b|^q \, dtds$$

$$= \int_a^{(a+b)/2} \int_{(a+b)/2}^b (s - t + b - a)^q \, dtds$$

$$= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ -(s - t + b - a)^{q+1} \Big|_{(a+b)/2}^b \right\} ds$$

(2.17)
$$= \frac{1}{q+1} \int_a^{(a+b)/2} \left\{ -(s - a)^{q+1} + \left(s + \frac{b - 3a}{2}\right)^{q+1} \right\} ds$$

$$= \frac{1}{(q+1)(q+2)} \left\{ -(s - a)^{q+2} + \left(s + \frac{b - 3a}{2}\right)^{q+2} \right\} \Big|_a^{(a+b)/2}$$

$$= \frac{(b - a)^{q+2} - 2 \left(\frac{b-a}{2}\right)^{q+2}}{(q+1)(q+2)},$$

(2.18)
$$I_3 = \int_{(a+b)/2}^b \int_a^{(a+b)/2} |t - s + b - a|^q \, dtds$$

$$= \int_{(a+b)/2}^b \int_a^{(a+b)/2} (t - s + b - a)^q \, dtdsds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^b \left\{ (t - s + b - a)^{q+1} \Big|_a^{(a+b)/2} \right\} ds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^b \left\{ \left(-s + \frac{3b - a}{2}\right)^{q+1} (b - s)^{q+1} \right\} ds$$

$$= \frac{1}{(q+1)(q+2)} \left\{ -\left(-s + \frac{3b - a}{2}\right)^{q+2} + (b - s)^{q+2} - \right\} \Big|_{(a+b)/2}^b$$

$$= \frac{(b - a)^{q+2} - 2 \left(\frac{b-a}{2}\right)^{q+2}}{(q+1)(q+2)}$$

$$I_4 = \int_{(a+b)/2}^b \int_{(a+b)/2}^b |t - s|^q \, dtds$$

$$= \int_{(a+b)/2}^b \left\{ \int_{(a+b)/2}^s (s - t)^q \, dt + \int_s^b (t - s)^q \, dt \right\} ds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^b \left\{ -(s - t)^{q+1} \Big|_{(a+b)/2}^s + (t - s)^{q+1} \Big|_s^b \right\} ds$$

$$= \frac{1}{q+1} \int_{(a+b)/2}^{b} \left\{ \left(s - \frac{a+b}{2} \right)^{q+1} + (b-s)^{q+1} \right\} ds$$

$$(2.19) \qquad = \frac{1}{(q+1)(q+2)} \left\{ \left(s - \frac{a+b}{2} \right)^{q+2} - (b-s)^{q+2} \right\} \Big|_{(a+b)/2}^{b}$$

$$= \frac{2 \left(\dfrac{b-a}{2} \right)^{q+2}}{(q+1)(q+2)}$$

Using (2.16)-(2.19) in (2.15), we get

$$(2.20) \qquad \left(\int_a^b \int_a^b |m(t) - m(s)|^q \, dt ds \right)^{\frac{1}{q}} = \left(\frac{2(b-a)^{q+2}}{(q+1)(q+2)} \right)^{\frac{1}{q}}$$

Similar way in (2.7), we have

$$(2.21) \qquad \left(\int_a^b \int_a^b |f'(t) - f'(s)|^p \, dt ds \right)^{\frac{1}{p}} \leq 2(b-a)^{\frac{1}{p}} \|f'\|_p.$$

Using (2.20) and (2.21) in (2.14), it follows that

$$(2.22) \qquad |M(f;a,b)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{\frac{1}{q}} \|f'\|_p$$

and similarly,

$$(2.23) \qquad |M(g;a,b)| \leq \left(\frac{2(b-a)}{(q+1)(q+2)} \right)^{\frac{1}{q}} \|g'\|_p.$$

Using (2.22) and (2.23) in (2.13), we obtain (2.10). $\square$

References

[1] S. S. Dragomir and S. Mabizela, Some error estimates in the trapezoidal quadrature rule, *RGIMA Res. Rep. Coll.*, **2** (5) (1999), 653-663.

[2] S. S. Dragomir, J. Sunde and C. Buşe, Some new inequalities for Jeffreys divergence measure in information, *RGIMA Res. Rep. Coll.*, **3** (2) (2000), 235-243.

[3] P. L. Čebyšev, Sur less expressions approximatives des integrales definies par les autres prises entre les memes limites, *Proc. Math. Soc. Charkov*, **2** (1882), 93-98.

[4] Z. Liu, Generalizations of some new Čebyšev type inequalities, *J. Inequal. Pure and Appl. Math.*, **8** (1) (2007), Art. 13.

[5] B. G. Pachpatte, New Čebyšev type inequalities via trapezoidal-like rules, *J. Inequal. Pure and Appl. Math.*, **7** (1) (2006), Art. 31.

[6] B. G. Pachpatte, On Čebyšev type inequalities involving functions whose derivatives belong to L_p spaces, *J. Inequal. Pure and Appl. Math.*, **7** (2) (2006), Art. 58.