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The Dual of a Strongly Prime Ideal

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Abstract. Let R be a commutative integral domain with quotient field K and let P be a nonzero strongly prime ideal of R. We give several characterizations of such ideals. It is shown that $(P : P)$ is a valuation domain with the unique maximal ideal P. We also study when P^{-1} is a ring. In fact, it is proved that $P^{-1} = (P : P)$ if and only if P is not invertible. Furthermore, if P is invertible, then $R = (P : P)$ and P is a principal ideal of R.

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1. Introduction

In this paper, we consider a commutative integral domain R in which a nonzero prime ideal P has the property that whenever P contains the product xy of two elements of the quotient field of R, then $x \in P$ or $y \in P$. Such prime ideals are called strongly prime ideal. In the second section of the paper, several characterizations of strongly prime ideals are given. The third section is concluded with a study of the dual of the strongly prime ideals of R.

Throughout this paper, R will be a commutative integral domain, K will denote its quotient field and I will be a nonzero ideal of R. The R-submodule

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J of K is called a fractional ideal if there exists an element $a \in R$ such that $aJ \subseteq R$. For a nonzero fractional ideal J of R, the fractional ideal $(R : J) = \{ x \in K \mid xJ \subseteq R \}$ is called the dual of J and is denoted by J^{-1}. In [8], Huckaba and Papick studied the question of when I^{-1} is a ring, and this question has received further attention in [1-4] and [6,7].

We note that while (I : I) is always an overring of R, I^{-1} need not be a ring at all. In fact, $(I : I)$ is the largest overring of R in which I is still an ideal. Clearly $(I : I) \subseteq I^{-1}$, and if we have equality, then I^{-1} is a ring. Example 3.1 of [1] shows that I^{-1} may be a ring strictly containing $(I : I)$. Our purpose here is to study P^{-1}, where P is a strongly prime ideal of R. We start by recalling the following results proved in [7] and [8].

Proposition 1.1. (See [8; Lemma 2.0]) *If I is a proper invertible ideal of R, then I^{-1} is not a subring of K.*

Proposition 1.2. (See [8; Proposition 2.3]) *Let $0 \neq P$ be a prime ideal of R. Then P^{-1} is a subring of K if and only if $P^{-1} = (P : P)$.*

Proposition 1.3. (See [8; Proposition 3.5]) *Let I be an ideal of a valuation domain R. Then I^{-1} is a subring of K if and only if I is a noninvertible prime ideal.*

Proposition 1.4. (See [7; Proposition 2.1]) *Let I be a nonzero ideal of R for which I^{-1} is a ring. Then P^{-1} is a ring for each minimal prime ideal of I.*

2. Strongly prime ideals

A prime ideal P of R is said to be *strongly prime*, if whenever $xy \in P$ for $x, y \in K$, then either $x \in P$ or $y \in P$. Strongly prime ideals were introduced by Hedstrom and Houston in [5], in their study of pseudo-valuation domains. In this section we give several characterizations and properties of such ideals.

Proposition 2.1. *Let R be an integral domain and P be a prime ideal of R. The following statements are equivalent:*

1. *P is a strongly prime ideal of R.*
2. *For each element $x \in K \setminus P$, $x^{-1}P \subseteq P$.*
3. *For every element $a \in R$, aP is comparable to every principal ideal of R.*
4. *For every element $a \in R$, aP is comparable to every ideal of R.*
5. *$P \subset Rx$, for every $x \in K \setminus P$.*
6. *If $P \subset Rx$ and $P \subset Ry$, for $x, y \in K$, then $P \subset Rxy$.*

Proof. 1 $\Rightarrow$ 2. We assume that $x \in K \setminus P$. Thus $x(x^{-1}P) = (xx^{-1})P = P$. Since P is strongly prime and $x \notin P$, we have $x^{-1}P \subseteq P$.
2 $\Rightarrow$ 3. Let $0 \neq a \in R$. For every element $x \in R$, if $\frac{x}{a} \in P$, then $x \in aP$ and so $Rx \subseteq aP$. If $\frac{x}{a} \notin P$, then $\frac{a}{x}P = (\frac{x}{a})^{-1}P \subseteq P$, by assumption. Hence $aP \subseteq Px \subseteq Rx$.
3 $\Rightarrow$ 4. Let I be an ideal of R and $a \in R$. If $I \nsubseteq aP$, then there exists an element $x \in I$ such that $x \notin aP$. Thus $aP \subseteq Rx \subseteq I$, by 3.
4 $\Rightarrow$ 5. Let $x \in K \setminus P$. Thus $x = \frac{a}{b}$, for some $a, b \in R$. If $bP \nsubseteq Ra$, then $Ra \subseteq bP$, by 4. Hence $x = \frac{a}{b} \in R\frac{a}{b} \subseteq P$, a contradiction. Therefore $bP \subseteq Ra$ and so $P \subseteq R\frac{a}{b} = Rx$. It is clear that $P \neq Rx$.
5 $\Rightarrow$ 6. Let $P \subset Rx$ and $P \subset Ry$, for some $x, y \in K$. If $xy \in P$, then $xy \in Ry$ and so $xy = ry$, for some $r \in R$. Hence $x = r \in R$. Similarly, we can show that $y \in R$. Since P is prime ideal and $xy \in P$, then $x \in P$ or $y \in P$. This is a contradiction. Therefore $xy \notin P$ and consequently $P \subset Rxy$, by assumption.
6 $\Rightarrow$ 1. It is obvious. □

It is clear that if P is a strongly prime ideal of R, then P is a strongly prime ideal of $(P : P)$. It follows from Proposition 2.1 that, P is a strongly prime ideal of R if and only if $x^{-1}P \subseteq P$, for every element $x \in K \setminus P$. Therefore $x^{-1} \in (P : P)$, for each $x \in K \setminus P$, and so we have:

Corollary 2.2. *Let R be an integral domain and P be a strongly prime ideal of R. Then $(P : P)$ is a valuation domain with maximal ideal P.* □

Corollary 2.3. *Every prime ideal contained in a strongly prime ideal of R is strongly prime.* □

Corollary 2.4. *Let R be an integral domain and P be a nonzero strongly prime ideal of R. Then we have:*

1. *For every element $x \in R \setminus P$, $P = xP$.*
2. *If S is a multiplicatively closed subset of R and $P \cap S = \emptyset$, then P is an ideal of R_S and $PR_S = P$.*

Proof. 1. Let $x \in R \setminus P$. Then $P \subset Rx$, by 5 of Proposition 2.1. Thus for each element $a \in P$, there exists $r \in R$ such that $rx = a \in P$ and so $r \in P$. Hence $P \subseteq xP \subseteq P$. Therefore $P = xP$.
2. Now, we assume that $\frac{a}{s} \in PR_S$, then $a \in P$ and so $a = rs$, for some $r \in P$. Therefore $\frac{a}{s} = \frac{rs}{s} = r \in P$. □

Definition 2.5. *A proper ideal I of R is called a divided ideal, if it is comparable to every ideal of R.*

We note that every strongly prime ideal of R is divided, by 4 of Proposition 2.1. Therefore we can conclude

Corollary 2.6. *Let R be an integral domain and P be a strongly prime ideal of R. Then*

1. *$P \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R.*
2. *If P is a minimal prime ideal of the ideal I of R, then $\sqrt{I} = P$. Hence P is the unique minimal prime ideal of I.* □

Now, we give another characterization of a strongly prime ideal in terms of properties of valuation domains. Before stating the next Proposition, we recall a result about valuation domains. The set of all ideals of a valuation domain are linearly ordered with respect to inclusion. Corollary 2.2 concludes the following:

Proposition 2.7. *Let R be an integral domain and P be an ideal of R. The following statements are equivalent:*

1. *P is a strongly prime ideal of R.*
2. *$(P : P)$ is a valuation domain with maximal ideal P.*
3. *P is a prime ideal of $(P : P)$ and the ideals of $(P : P)$ are linearly ordered.*
4. *P is a prime ideal of $(P : P)$ and every principal ideal of $(P : P)$ is a divided ideal.*
5. *There exists a valuation domain T containing R such that P is a prime ideal of T.* □

The following is an example of a prime ideal P of R such that $(P : P)$ is a valuation domain, but P is not a prime ideal of $(P : P)$. Therefore P is not a strongly prime ideal of R.

Example 2.8 Let $R = Q[[x^2, x^3]]$. Then R is a local ring with maximal ideal $P =< x^2, x^3 >$. It is clear that $P^{-1} = (P : P) = Q[[x]]$ is a valuation domain, but P is not a prime ideal of $(P : P)$, because $x \notin P$ and $x^2 \in P$.

We can now prove a result which shows that an invertible strongly prime ideal is a principal maximal ideal.

Theorem 2.9. *Let R be an integral domain and P be a nonzero strongly prime ideal of R. If P is invertible, then R is a valuation domain with maximal ideal P. Furthermore, P is a principal ideal.*

Proof. Since P is invertible, $PP^{-1} = R$ and so $1 = \sum_{i=1}^{n} a_i b_i$, for some elements $a_1, a_2, \cdots, a_n$ of P and $b_1, b_2, \cdots, b_n$ of P^{-1}. For every element $a \in R \setminus P$, we have $P = aP$, by 1 of Corollary 2.4. Thus, for each i, $(1 \leq i \leq n)$, there exists an element $s_i \in P$ such that $as_i = a_i$. Hence,

$$1 = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} as_i b_i = a(\sum_{i=1}^{n} s_i b_i).$$

Since $s_i b_i \in PP^{-1} = R$, for all i, a is an unit of R. Therefore P is the unique maximal ideal of R and consequently R is quasi-local.

For every element $x \in K \setminus R$, $x^{-1}P \subseteq P$, by 2 of Proposition 2.1, and so $x^{-1}a_i \in P$, for all i. Thus $x^{-1}a_i b_i \in PP^{-1} = R$, for all i. Hence $x^{-1} = x^{-1}(\sum_{i=1}^{n} a_i b_i) = \sum_{i=1}^{n} x^{-1} a_i b_i \in R$. Therefore R is a valuation domain and by Corollary 2.2, $R = (P : P)$. On the other hand, since P is an invertible ideal of a quasi-local ring, P is a principal ideal. $\square$

3. Dual of a strongly prime ideal

In general, a prime ideal P of R need not be a prime ideal of $(P : P)$, (see [8; Example 2.5]). In [8; Proposition 2.4] it is proved that if P^{-1} is a ring, for a prime ideal P which is not divisorial, then P is a prime ideal of P^{-1}. In this section, we show that if $0 \neq P$ is a strongly prime ideal of R which is not maximal, then $P^{-1} = (P : P) = R_P$ is a valuation domain with the maximal ideal P.

Theorem 3.1. *Let R be an integral domain and M be a strongly prime ideal of R. For every prime ideal P, if $P \subset M$, then $(P : P) = R_P$ is a valuation domain with maximal ideal P. Furthermore, if R is Noetherian, then $dim R_P \leq 1$.*

Proof. By Corollary 2.3, P is a strongly prime ideal and so $T = (P : P)$ is a valuation domain with the unique maximal ideal P, by Corollary 2.2. We now prove that $T = R_P$. Let $x \in R \setminus P$. Then $x^{-1} \in T$, by 2 of Proposition 2.1, and consequently $R_P \subseteq T$.

Conversely, let $x \in T$. If $x \in R$, then $x \in R_P$ and if $x \notin R$, then $x^{-1} \in T \setminus P$. Now, if $x^{-1} \in R$, then $x = \frac{1}{x^{-1}} \in R_P$. If $x^{-1} \notin R$, consider an element $s \in M \setminus P$, we have $(sx)x^{-1} = s \in M$. Since M is a strongly prime ideal and $x^{-1} \notin M$, $a = sx \in M \subseteq R$. Therefore $x = \frac{a}{s} \in R_P$ and consequently

$(P : P) = R_P$. Now, if R is Noetherian, then P is a principal ideal of R_P, by [9; Theorem 5.9], and so $dim R_P = ht P \leq 1$, by Krull's principal ideal Theorem [10; Theorem 15.2]. □

The next result is a generalization of Proposition 1.3 and [8; Corollary 3.6]. Example 2.8 shows that P^{-1} may be a valuation domain, but P is not a strongly prime ideal.

Theorem 3.2. *If P is a nonzero strongly prime ideal of R, then P^{-1} is a valuation domain if and only if P is a noninvertible ideal. Furthermore, in this case, we have $P^{-1} = (P : P)$ and P is the unique maximal ideal of P^{-1}.*

Proof. If P^{-1} is a ring, then P is not invertible, by Proposition 1.1. We now assume that P is not invertible. By Corollary 2.2, it is enough to prove that $PP^{-1} \subseteq P$. Let $a \in P$ and $b \in P^{-1}$. If $ab \notin P$, then $(ab)^{-1} \in (P : P)$, by 2 of Proposition 2.1. Hence $b^{-1} = a(ab)^{-1} \in P$, and so $1 = b^{-1}b \in PP^{-1}$. Therefore $PP^{-1} = R$. This contradicts the fact that P is not invertible. □

Corollary 3.3. *Let P be a nonzero strongly prime ideal of R. If P is a maximal ideal of R, then either P is an invertible ideal or P^{-1} is a ring.* □

An integral domain R is called a *pseudo-valuation domain*, if each prime ideal of R is strongly prime. Obviously, every valuation domain is a pseudo-valuation domain, but there exists a pseudo-valuation domain which is not a valuation domain, (see [5; Example 2.1 and Example 3.6]). By 4 of Proposition 2.1, it is clear that every pseudo-valuation domain R is quasi-local and so each invertible ideal of R is principal. Therefore, Theorem 3.1 and Corollary 3.2 imply that

Corollary 3.4. *Let R be a pseudo-valuation domain with the unique maximal ideal M. Then for every nonzero prime ideal $P \neq M$ of R, $P^{-1} = (P : P) = R_P$ is a valuation domain with maximal ideal P. Furthermore, if M is not principal, then M^{-1} is a ring and $M^{-1} = (M : M)$.* □

Proposition 3.5. *Let R be an integral domain and P be a nonzero strongly prime ideal of R. If $Q = \bigcap_{n=1}^{\infty} P^n$, then*

1. *Q is a strongly prime ideal of R.*
2. *If $P \neq P^2$, then either $Q = \{0\}$ or Q^{-1} is a valuation domain.*

Proof. 1. It is clear that Q is an ideal of R. Let $T = (P : P)$. By Corollary 2.2, T is a valuation domain with maximal ideal P. Thus for every $a \in T$, $aP \subseteq P$ and so $aP^n \subseteq P^n$, for each integer n. Hence P^n is an ideal of T, for all n, and so Q is an ideal of T. Moreover, P^n is a divided ideal of T. Now we assume that $x, y \in T$ and $xy \in Q$. If $x \notin Q$, then $x \notin P^n$ for some integer n. Thus $P^n \subset Tx$. Hence for each integer m, we have

$$xy \in Txy \subseteq Q \subseteq P^{m+n} = P^m P^n \subseteq P^m x$$

and so $y \in P^m$. Therefore $y \in Q$ and so Q is a prime ideal of T. Hence Q is a prime ideal of R. Since $Q \subseteq P$ and P is a strongly prime ideal of R, Q is a strongly prime ideal of R, by Corollary 2.3.
2. Let $Q \neq \{0\}$. Since $P \neq P^2$, we have $Q \subset P$. Then Q is not maximal and so Q is not invertible, by Theorem 2.9. Therefore Q^{-1} is a valuation domain, by Theorem 3.2. □

Theorem 3.6. *Let R be an integral domain and P be a minimal prime ideal of the ideal I of R. Suppose also that P is a strongly prime ideal of R. Then we have:*

1. $(I : I) \subseteq (P : P)$.
2. *If for each $x \in I^{-1}$, $x^2 \in I^{-1}$, then $I^{-1} = (P : I)$.*
3. *If I^{-1} is a ring, then $I^{-1} = P^{-1} = (P : P) = (P : I)$.*

Proof. 1. Let $a \in (I : I)$ and $b \in P$. Then by 2 of Corollary 2.6, $b^n \in I$, for some integer n, also, we have $a^n \in (I : I)$. Hence $(ab)^n = a^n b^n \in I \subseteq P$. Since P is a strongly prime ideal, $ab \in P$ and so $aP \subseteq P$. Therefore $(I : I) \subseteq (P : P)$.
2. It is clear that $(P : I) \subseteq (R : I) = I^{-1}$. We now assume that $a \in I^{-1}$. Thus $a^2 \in I^{-1}$ and so $a^2 I \subseteq R$. For every element $x \in I$, we have $a^2 x \in R$. Hence $(ax)^2 = a^2x^2 = (a^2x)x \in I \subseteq P$, and consequently $ax \in P$. Therefore $aI \subseteq P$. Then $a \in (P : I)$.
3. Since I^{-1} is a ring, P^{-1} also is a ring, by Proposition 1.4. Then $P^{-1} = (P : P)$, by Proposition 1.2. Moreover, by 2, $I^{-1} = (P : I)$. Now, if $a \in (P : I)$ then $a^m \in (P : I)$, for all m, because I^{-1} is a ring, and so $a^m I \subseteq P$, for each m. For every $b \in P$, we have $b^n \in I$, for some integer n. Hence $(ab)^n = a^n b^n \in a^n I \subseteq P$ and consequently $ab \in P$, because P is a strongly prime ideal. Thus $a \in (P : P)$. Therefore $I^{-1} = (P : I) \subseteq (P : P) = P^{-1}$. On the other hand, $(P : P) \subseteq (P : I)$, because $I \subseteq P$. □

We recall that, every ideal of a Dedekind domain is an invertible ideal. Therefore, the next result follows from Theorem 2.8 and Corollary 2.3.

Proposition 3.7. *Let R be a Dedekind domain and P a nonzero strongly prime ideal of R. Then R is a valuation domain with the unique prime ideal P.* □

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