Iranian Journal of Mathematical Sciences and Informatics
Vol. 5, No. 1 (2010), pp. 7-18
DOI: 10.7508/ijmsi.2010.01.002

Clifford Wavelets and Clifford-valued MRAs

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Abstract. In this paper using the Clifford algebra over $\mathbb{R}^4$ and its matrix representation, we construct Clifford scaling functions and Clifford wavelets. Then we compute related mask functions and filters, which arise in many applications such as quantum mechanics.

Keywords: Clifford Wavelets, Clifford algebra, Multiresolution Analysis, Wavelets.

2000 Mathematics subject classification: 42C15, 46E15, 50C20, 42B99, 42C05.

1. Introduction

A complex-valued representation of a real 1-dimensional signal is an important tool in analysis of signal processing. The reason is that in its polar representation, the modulus of the complex signal is identified as a local quantitative measure of a signal, called local amplitude, and the argument of the complex signal is identified as a local measure for the qualitative information of a signal, called local phase. First step for generalizing such representation system was quaternion-valued representation, on which a signal can be expressed by four parameters as its local quantitative measures.

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Received 18 October 2009; Accepted 15 March 2010

On the other hand wavelets are a very useful and wide applied tools for practical applications in signal and image processing, multi-satellite measurements of electromagnetic wave fields, analysis of climate-related time-series and analysis space weather effects and so on. One usual way to construct wavelets pass through multiresolution analysis (MRA), which is a procedure for constructing wavelets from a scaling function. Now if the scaling function is a matrix of functions, we deal with matrix-valued MRAs. In this paper we show that any real or complex Clifford algebra can be identified with a suitable matrix algebra, then via this representation, Clifford-valued scaling functions, Clifford-valued MRAs and Clifford wavelets are given.

Notations. For an algebra $\mathbb{K}$, we denote its product with ".". $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are algebra of real numbers, complex numbers and quaternions, respectively. $\mathbb{K}[n]$ is the algebra of $n \times n$ matrices over field $\mathbb{K}$. $\otimes_{\mathbb{K}}$ denotes tensor product over field $\mathbb{K}$.

This paper is organized as follow: in second section we introduce the n-dimensional Clifford algebra (on brief) and some useful theorems on it, then we discuss the $Cl(\mathbb{R}^4)$ and $\mathbb{C}l(\mathbb{R}^4)$ (real and complex forms of Clifford algebra on $\mathbb{R}^4$, resp.) and their matrix representations. Section 3 consists of multiresolition analysis (MRA) and Clifford wavelet structures. In section 4, we compute Clifford wavelets matrices on $\mathbb{R}^4$.

2. Clifford Algebra

In this section we mention some definitions and basic facts about Clifford algebras.

Definition 2.1. let V be a finite dimensional vector space on the field $\mathbb{F}$. A quadratic form (q-form) on V is a function $h : V \times V \longrightarrow \mathbb{F}$, such that
$h(\alpha x_1 + x_2, y) = \alpha h(x_1, y) + h(x_2, y)$
$h(x, \alpha y_1 + y_2) = \alpha h(x, y_1) + h(x, y_2)$.

Furthermore if $h(x, y) = h(y, x)$ then h is called symmetric. For any q-form h, there exists a matrix representation $A = (A_{ij})$ such that $A_{ij} = h(e_i, e_j)$ where $\{e_1, e_2, \cdots, e_n\}$ is a basis for V. The q-form h is called nondegenerate, if $det(h(e_i, e_j)) \neq 0$.
Let V be an n-dimensional vector space on the field F, and h be a non-degenerate symmetric q-form on V, then there exists an ordered basis $B = \{e_1, e_2, \cdots, e_n\}$ for V such that $A = (A_{ij})$ is diagonal. In particular for $\mathbb{F} = \mathbb{R}$

$$h(e_i, e_j) = \begin{cases} \pm 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If the matrix A have p-times 1 and q-times -1 on its diameter such that $p + q = n$, then h will be shown with $h(p, q)$. For h, a nondegenerate q-form on

real vector space V, the pair (V, h) is called a quadratic space(q-space). For describing the Clifford algebra on vector space V, consider the commutative tensor algebra $T(V) = \bigoplus_{r=0}^{\infty} \otimes^r V$ on real q-space (V, h) with unit 1. Let $I_h(V) = \langle V \otimes V + h(V, V) \rangle$ then I_h is a two-sided ideal in $T(V)$. The quotient space $\frac{T(V)}{I_h(V)}$ is called the *Clifford algebra* on V and is denoted by $Cl(V, h)$. The induced product, from tensor product on T(V), is called *Clifford product* and will be shown with ".", $(Cl(V, h), ".")$ is again a commutative algebra with unit. If h is $h(p, q)$ then $Cl(V, h)$ will be shown by $Cl(p, q)$.
By considering the canonical projection map $\pi_h : T(V) \longrightarrow Cl(V, h)$, one can find that the map $\theta_V : V \longrightarrow Cl(V, h)$ is one-to-one. This fact says that $Cl(V, h)$ is generated by vector space $V \subset Cl(V, h)$ and identity 1, and its product satisfies the following relations:
1) $v \cdot v = -h(v, v)1 \quad$ for any $v \in Cl(V, h)$
2) $v \cdot w + w \cdot v = -2h(v, w)$.
In view of previous equations we can obtain the universal map for Clifford algebras as follow:

Proposition 2.1. *Let $\mathcal{A}$ be a commutative* $\mathbb{K}$-Algebra *with unit 1, and $f : V \longrightarrow \mathcal{A}$ be a linear map such that: $f(v) \cdot f(v) = -h(v, v)1$ for any $v \in V$, then f can be uniquely extended to the algebraic homomorphism $\widetilde{f} : Cl(V, h) \longrightarrow \mathcal{A}$. Furthermore, $Cl(V, h)$ is the unique associated* $\mathbb{K}$-Algebra *with this property.*

In other word if (V, h) is a q-space, then there exists a Clifford algebra associated to it and is unique up to an isomorphism. This is easy to show that if $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis for real vector space V, then the set $\{1, e_i, e_ie_j, e_ie_je_k, \cdots, e_1e_2e_3 \cdots e_n : i+1 = j, j+1 = k\}$ is a basis for $Cl(V, h)$. Note that $Cl(V, h) = \frac{T(V)}{I_h} = \frac{R \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \cdots}{\langle V \otimes V + h(V)1 \rangle}$, and

$$T(V) = a_0 + \sum_{i=1}^{n} a_i e_i + \sum a_{ij} e_i \otimes e_j + \sum a_{ijk} e_i \otimes e_j \otimes e_k + \cdots + a_{i_1 \ldots i_n} e_1 \otimes e_2 \otimes \ldots \otimes e_n.$$

Also $V \otimes V + h(V)1 = 0 \quad$ implies that $\quad V \otimes V = -h(V)1$.

Example 2.2. *Let $V = R^2$, and h be the quadratic form obtained by the matrix*

$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e $V = \mathbb{R}^2 = \langle e_1, e_2 \rangle$. Dim$V = 2$, so dim$Cl(V) = 4$ and

$$\begin{aligned} Cl(V) &= Cl(\mathbb{R}^2) = \langle 1, e_1, e_2, e_1e_2 \rangle \\ &= \{a_0 + a_1e_1 + a_2e_2 + a_{12}e_1e_2 : {e_1}^2 = {e_2}^2 = -1, e_1 \cdot e_2 = -e_2 \cdot e_1\} \end{aligned}$$

where $(e_1 \cdot e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = (-1)(-1)(-1) = -1$.
So if we define $\psi : Cl(\mathbb{R}^2) \longrightarrow \mathbb{H}$ by

$$\psi(1) = 1, \psi(e_1) = i, \psi(e_2) = j, \psi(e_1e_2) = \psi(e_3) = k$$

then, since ψ is an algebraic homomorphism, $Cl(\mathbb{R}^2) \cong \mathbb{H}$.

There are useful algebraic isomorphisms for $Cl(p,q)$ such as

$$Cl(n,0)\otimes Cl(0,2)\cong Cl(0,n+2) \tag{2.1}$$

$$Cl(0,n)\otimes Cl(2,0)\cong Cl(n+2,0)$$

$$Cl(p,q)\otimes Cl(1,1)\cong Cl(p+1,q+1),$$

where $n,p,q\geq 0$ such that $n=p+q$.
Now we introduce a useful tool. Complexification is one of the important tools in linear algebra which make it more flexible. Let (V,h) be a real q-space. The complexification of V is the vector space $W=V\otimes_{\mathbb{C}}\mathbb{C}$ such that for $w\in W: w=v\otimes\lambda=v\otimes(a+ib)=v\otimes a+v\otimes ib=1\otimes av+i(1\otimes bv)$. This means that any element of W can be written as $x+iy$ where $x,y\in V$. Now let g be a nondegenerate q-form on V. Then $g_W: W\times W\longrightarrow\mathbb{C}$ is a nondegenerate q-form on $W=V\otimes\mathbb{C}$ defined by $g_W(x\otimes\lambda,y\otimes\gamma)=\lambda\gamma g(x,y)$. From this point of view the complexification of $Cl(V)$ is $Cl(V)\otimes\mathbb{C}$ and if $W=V\otimes_{\mathbb{C}}\mathbb{C}$ then $Cl(W)=Cl(V)\otimes_{\mathbb{R}}\mathbb{C}$.

Lemma 2.3. *Let V be a real n-dimensional vector space, then*

$$\mathbb{C}l(V\oplus\mathbb{R}^2)\otimes\mathbb{C}\cong(Cl(V)\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}(Cl(\mathbb{R}^2)\otimes\mathbb{C}).$$

Proof. Let $\{\nu_1,\cdots,\nu_n\}$ be an orthonormal basis for V and $\{e_1,e_2\}$ be the standard basis for $\mathbb{R}^2$. Consider the real map $\theta: V\oplus\mathbb{R}^2\longrightarrow(Cl(V)\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}(Cl(\mathbb{R}^2)\otimes\mathbb{C})$ defined by

$$(\nu_j,0)\longmapsto i\nu_j\otimes e_1e_2,\quad 1\leq j\leq n,\ \ (0,e_r)\longmapsto 1\otimes e_r\ \ r=1,2.$$

so θ extends to algebra homomorphism $\mathbb{C}l(V\oplus\mathbb{R}^2)\otimes\mathbb{C}\cong(Cl(V)\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}(Cl(\mathbb{R}^2)\otimes\mathbb{C})$. On the other hand domain and range of θ have the same dimension and it is onto, so θ is isometry. □

the following lemma is the key tool for describing the complex Clifford algebras.

Lemma 2.4. *Let V be a real vector space such that $dimV=2n$, then $Cl(V)\otimes_{\mathbb{R}}\mathbb{C}$ is isomorphic to the matrix algebra $\mathbb{C}[2^n]$.*
If $dimV=2n+1$ then $Cl(V)\otimes_{\mathbb{R}}\mathbb{C}$ is isomorphic to $\mathbb{C}[2^n]\oplus\mathbb{C}[2^n]$.

Proof. We refer interested reader to [2], for an extended proof. □

2.1. **Construction of Clifford Algebra on $\mathbb{R}^4$.** Now we are going to show that for $V=\mathbb{R}^4$, $Cl(V)$ is $\mathbb{H}[2]\cong\mathbb{C}[4]$. We know that, via the algebraic isomorphism

$$a+bi+cj+dk\longmapsto\begin{pmatrix} a+id & b+ic \\ -b+ic & a-id\end{pmatrix},$$

$\mathbb{H}$ is isomorphic to $\mathbb{C}[2]$. Now if $V = \mathbb{R}^4 = \langle e_0, e_1, e_2, e_3\rangle$ with Riemannian form $h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ on it, then

$$Cl(\mathbb{R}^4) = \{a_0 + \sum_{i=1}^{4} a_i e_i + \sum_{i<j} a_{ij} e_i e_j + \sum_{i<j<k} a_{ijk} e_i e_j e_k + a_{1234} e_1 e_2 e_3 e_4 : e_i e_j = -e_j e_i, {e_i}^2 = -1, a_i \in \mathbb{R}\}$$

this means that $Cl(\mathbb{R}^4)$ is spanned by $2^4 = 16$ vectors:

$$1, E_1, E_2, E_3, E_4, E_1E_2, E_1E_3, E_1E_4, E_2E_3, E_2E_4, E_3E_4,$$

$$E_1E_2E_3, E_1E_2E_4, E_1E_3E_4, E_2E_3E_4, E_1E_2E_3E_4,$$

as a basis. On the other hand

$$Cl(0,2) = Cl(\mathbb{R}^2, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) = \langle e_0, e_1, e_2, e_3 = e_1e_2\rangle$$

where $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ such that ${e_0}^2 = {e_1}^2 = {e_2}^2 = 1, (e_1e_2)^2 = -1$ and

$$Cl(2,0) = Cl(\mathbb{R}^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \cong \mathbb{H} = \langle {e_0}', {e_1}', {e_2}', {e_3}' = {e_1}'{e_2}'\rangle$$

where ${e_0}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, {e_1}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, {e_2}' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, {e_3}' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.
Now if in (2.1) we set $n = 2$ then

$$Cl(0,2) \otimes Cl(2,0) \cong Cl(4,0).$$

Through the relation $A \otimes B = (A_{ij}B)$ between matrices we can find the matrix representation for $Cl(4,0)$'s bases:
$E_0 = e_0 \otimes {e_0}' = I,\ E_1 = e_0 \otimes {e_3}',\ E_2 = e_2 \otimes {e_1}',\ E_3 = e_1 \otimes {e_1}', E_4 = e_0 \otimes {e_2}',\ \ E_1E_2 = e_2 \otimes {e_2}',\ \ E_1E_3 = e_1 \otimes {e_2}',\ \ E_1E_4 = -(e_0 \otimes {e_1}'),\ \ E_2E_3 = e_3 \otimes {e_0}',\ \ E_2E_4 = e_1 \otimes {e_3}',\ \ E_3E_4 = e_2 \otimes {e_3}',\ \ E_1E_2E_3 = e_3 \otimes {e_3}',\ \ E_1E_2E_4 = -(e_2 \otimes {e_0}'),\ \ E_2E_3E_4 = e_3 \otimes {e_2}',\ \ E_1E_3E_4 = -(e_1 \otimes {e_0}'), E_1E_2E_3E_4 = -(e_3 \otimes {e_1}')$.
This means that for any $\rho \in Cl(\mathbb{R}^4)$ we have

$$\rho = a_0 + a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 + a_{12}E_1E_2 + a_{13}E_1E_3 + a_{14}E_1E_4 + a_{24}E_2E_4$$

$$+ a_{34}E_3E_4 + a_{23}E_2E_3 + a_{123}E_1E_2E_3 + a_{124}E_1E_2E_4$$

$$+ a_{234}E_2E_3E_4 + a_{134}E_1E_3E_4 + a_{1234}E_1E_2E_3E_4$$

By the above matrix representation for E_i's, associated matrix to ρ is:

$$\begin{pmatrix} a_0+a_1i+a_{34}i- & a_2+a_4i+a_{12}i- & -a_{24}i-a_{23}-a_{123}i- & a_3+a_{13}ia_{234}i+ \\ a_{124} & a_{14} & a_{134} & a_{1234} \\ -a_2+a_4i+a_{12}i+ & a_0-a_1i-a_{34}i- & -a_3+a_{13}i-a_{234}i- & -a_{23}+a_{24}i+a_{123}i- \\ a_{14} & a_{124} & a_{1234} & a_{134} \\ a_{23}+a_{24}i+a_{123}i- & a_3+a_{13}i+a_{234}i- & a_0+a_1i-a_{34}i+ & -a_2+a_4i-a_{12}i- \\ a_{134} & a_{1234} & a_{124} & a_{14} \\ -a_3+a_{13}i+a_{234}i+ & a_{23}-a_{24}i-a_{123}i- & a_2+a_4i-a_{12}i+ & a_0-a_1i+a_{34}i+ \\ a_{1234} & a_{134}a & a_{14} & a_{124} \end{pmatrix}.$$

Now if we set
$A_1 = a_0 + ia_1$, $B_1 = -a_{124} + ia_{34}$, $A_2 = a_2 + ia_4$, $B_2 = a_{14} + ia_{12}$, $A_3 = a_{23} + ia_{24}$, $B_3 = -a_{134} + ia_{123}$, $A_4 = a_3 + ia_{13}$, $B_4 = a1234 + ia_{234}$,
and then set $A = A_1 + B_1$, $B = A_1 - B_1$, $C = A_2 - \overline{B_2}$, $D = A_2 + \overline{B_2}$, $E = A_3 + B_3$, $F = -A_3 + B_3$, $G = A_4 + \overline{B_4}$, $H = A_4 - \overline{B_4}$, ρ can be shown as

$$\rho \cong \begin{pmatrix} A & -C & F & -G \\ \overline{C} & \overline{A} & \overline{G} & \overline{F} \\ E & -H & B & -D \\ \overline{H} & \overline{E} & \overline{D} & \overline{B} \end{pmatrix} := M_Q, \tag{2.2}$$

A simpler representation for ρ is $\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \lambda \end{pmatrix}$, which is a $2 \times 2-$matrix in $\mathbb{H}$, with $\alpha = A - j\overline{C}$, $\beta = F - j\overline{G}$, $\gamma = E - j\overline{H}$, $\lambda = B - j\overline{D}$.
Till now we've found the matrix representations for $Cl(\mathbb{R}^4)$ such that $\mathbb{H}[2] \cong \mathbb{C}[4]$. By considering the complexification of $Cl(\mathbb{R}^4)$ we will work with $\mathbb{C}[4]$, which is a more general and flexible case.
Let M_Q be the set of all $4 \times 4-$matrices in $\mathbb{C}[4]$ which are like above then M_Q excepting the zero matrix is a subgroup of $GL(2, \mathbb{C})$ in the sense of matrix multiplication.
In next step we generalize these concepts to an MRA.

3. $\mathbb{C}l(\mathbb{R}^4)$-valued MRA

3.1. **General construction and mask functions.** Let $L^2(\mathbb{R}, \mathbb{C}[r]) = \{\mathbf{F}(t) = (F_{m,n}(t)) : t \in \mathbb{R}, F_{m,n} \in L^2(\mathbb{R}), 1 \leq m, n \leq r\}$ be the space of matrix-valued functions defined on $\mathbb{R}$ with values in $\mathbb{C}[r]$. The norm on $L^2(\mathbb{R}, \mathbb{C}[r])$ is the Ferobenious norm : $\| \mathbf{F}(t) \| = [\sum_{m,n} \int_{\mathbb{R}} |F_{m,n}(t)|^2 dt]^{\frac{1}{2}}$ and for $\mathbf{F}, \mathbf{G} \in L^2(\mathbb{R}, \mathbb{C}[r])$, the "inner product" is defined by $\langle \mathbf{F}, \mathbf{G} \rangle_{L^2(\mathbb{R}, \mathbb{C}(r))} := \int_{\mathbb{R}} \mathbf{F}(t) \mathbf{G}^{\dagger}(t) dt$ where $\mathbf{G}^{\dagger}$ is the complex conjugate transpose of $\mathbf{G}$. As pointed out in [7] and [8] such operation, which is an integral of matrix product, is not really an inner product but it has the linear and commutative properties:
1.$\langle \mathbf{F}_1, a\mathbf{F}_2 + b\mathbf{F}_3 \rangle = a^{\dagger} \langle \mathbf{F}_1, \mathbf{F}_2 \rangle + b^{\dagger} \langle \mathbf{F}_1, \mathbf{F}_3 \rangle$
2.$\langle \mathbf{F}_1, \mathbf{F}_2 \rangle = \langle \mathbf{F}_2, \mathbf{F}_1 \rangle^{\dagger}$.

Here the orthogonality of $\mathbf{F}_j$ and $\mathbf{F}_k$ is identified with $\langle \mathbf{F}_j, \mathbf{F}_k \rangle = I_r \delta_{jk}$ where I_r is identity matrix and δ_{jk} the Kronecker delta. Now let $\mathbf{X}(t)$ be a $\mathbb{C}l(\mathbb{R}^4)$-valued function. Then $\mathbf{X}(t)$ via its components has a representation like M_Q, as shown in (2.2) and matrix representation of $\mathbf{X}(t)$ is shown with $M_Q(\mathbf{X})$. Define $L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4]) = \{M_Q(\mathbf{X}) : x_{ij} \in L^2(\mathbb{R}), 1 \le i,j \le 4\} \subseteq L^2(\mathbb{R}, \mathbb{C}[4])$, and

$$L^2(\mathbb{R}, \mathbb{C}l(\mathbb{R}^4)) = \{\mathbf{X}(t) = x_0(t) + x_1(t)E_1 + \ldots + x_{1234}(t)E_{1234} : x_i \in L^2(\mathbb{R})\},$$

then we can identify $L^2(\mathbb{R}, \mathbb{C}l(\mathbb{R}^4))$ with $L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])$ by $T : L^2(\mathbb{R}, \mathbb{C}l(\mathbb{R}^4)) \longrightarrow L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])$ such that

$$\mathbf{X}(t) \longmapsto \begin{pmatrix} x_A & -x_C & x_F & -x_G \\ \overline{x}_C & \overline{x}_A & \overline{x}_G & \overline{x}_F \\ x_E & -x_H & x_B & -x_D \\ \overline{x}_H & \overline{x}_E & \overline{x}_D & \overline{x}_B \end{pmatrix} = M_Q(\mathbf{X}),$$

where $x_A = x_0(t) + ix_1(t) + ix_{34}(t) - x_{124}(t)$ and all other entries are similar to M_Q's entries.

Immediately we realize that $\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2(\mathbb{R}, \mathbb{C}l(\mathbb{R}^4))} \longmapsto \langle M_Q(\mathbf{X}), M_Q(\mathbf{Y}) \rangle_{L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])}$, where $\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2(\mathbb{R}, \mathbb{C}l(\mathbb{R}^4))} = \int_{\mathbb{R}} \mathbf{X}\mathbf{Y}^\dagger dt$.

Now by considering $\mathbb{C}l(\mathbb{R}^4) \cong \mathbb{C}[4]$, we will investigate some results in matrix-valued MRAs.

Definition 3.1. The matrix-valued function $\Phi(t) = (\varphi_{m,n}(t))_{r \times r} \in L^2(\mathbb{R}, \mathbb{C}[r])$ generates a matrix-valued multiresolution analysis for $L^2(\mathbb{R}, \mathbb{C}[r])$ if the subspaces $\mathbf{V}_j = span\{2^{\frac{j}{2}} \Phi(2^j t - k) : k \in \mathbb{Z}\}$ are nested: $\cdots \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \cdots$, and the following conditions hold:

1) $\overline{\bigcup_{j \in \mathbb{Z}} \mathbf{V}_j} = L^2(\mathbb{R}, \mathbb{C}[r])$,

2) $\bigcap \mathbf{V}_j = 0_r$, in which 0_r is the $r \times r$-zero matrix.

3) $\mathbf{X}(t) \in \mathbf{V}_0 \Longleftrightarrow \mathbf{X}(2^j t) \in \mathbf{V}_j, \quad j \in \mathbb{Z}$,

4) $\mathbf{X}(t) \in \mathbf{V}_0 \Longleftrightarrow \mathbf{X}(t-k) \in \mathbf{V}_0, \quad k \in \mathbb{Z}$,

5) $\{\Phi(t-k) : k \in \mathbb{Z}\}$ form an orthonormal basis for $\mathbf{V}_0$.

Remark 3.1. *: A sequence* $\{\Phi_k\}_{k \in \mathbb{Z}}$*in* $L^2(\mathbb{R}, \mathbb{C}(r))$ *is called an orthonormal basis if it is an orthonormal set,* $\langle \Phi_j, \Phi_k \rangle = \boldsymbol{I}_r \delta_{jk}$*, and for any* $\boldsymbol{X}(t) \in L^2(\mathbb{R}, \mathbb{C}[r])$ *there exists constant matrix-sequence* $\{\boldsymbol{A}_k\}_{k \in \mathbb{Z}}$ *such that* $\boldsymbol{X}(t) = \sum_{k \in \mathbb{Z}} \boldsymbol{A}_k \Phi_k(t)$.

Condition (5) means that $X(t) = \sum_{k \in \mathbb{Z}} \mathbf{A}_k \Phi_k(t-k)$, which Ferobenious norm will guarantee the convergence of infinite sum, and $\mathbf{A}_k = \langle X, \Phi_k(t-k) \rangle$ by orthonormality. Also since $\Phi(t) \in \mathbf{V}_0 \subset \mathbf{V}_1$, then the two-scale matrix dilation equation is

$$\Phi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} \mathbf{G}_k \Phi(t-k) \tag{3.1}$$

which combined with orthonormality of Φ's means

$$\sum_{k\in\mathbb{Z}} \mathbf{G}_k \mathbf{G}^{\dagger}_{2l+k} = \mathbf{I}_r \delta_{l0}, \quad l \in \mathbb{Z} \quad . \tag{3.2}$$

Let $\widehat{\mathbf{G}}(f) = \sum_{k\in\mathbb{Z}} \mathbf{G}_k e^{-2\pi i k f}$ be the matrix mask function, then (3.2) implies that

$$\widehat{\mathbf{G}}(f)\widehat{\mathbf{G}}^{\dagger}(f) + \widehat{\mathbf{G}}(f+\frac{1}{2})\widehat{\mathbf{G}}^{\dagger}(f+\frac{1}{2}) = 2I_r, \tag{3.3}$$

Define matrix Fourier transform for $\Phi(t)$ by $\widehat{\Phi}(f) := \int_{\mathbb{R}} \Phi(t)e^{-2\pi i k f t}dt$. Then (3.1) gives $\widehat{\Phi}(f) = \frac{1}{\sqrt{2}}\widehat{\mathbf{G}}(\frac{f}{2})\widehat{\phi}(\frac{f}{2})$, where by setting $f = 0$ we get $\widehat{\mathbf{G}}(0) = \sum \mathbf{G}_k = \sqrt{2}\mathbf{I}_r$, $\widehat{\mathbf{G}}(\frac{1}{2}) = 0$. Define the function matrix $\Psi(t) = (\psi_{m,n}(t))_{r\times r} \in L^2(\mathbb{R}, \mathbb{C}[r])$ and corresponding subspace $\mathbf{W}_j = span\{2^{\frac{j}{2}}\Psi(2^j t - k) : k \in \mathbb{Z}\}$. $\mathbf{W}_j$ is orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$ i.e. $\mathbf{V}_{j+1} = \mathbf{V}_j \oplus \mathbf{W}_j$, $\mathbf{V}_j \perp \mathbf{W}_j$ and $\bigoplus_{j\in\mathbb{Z}} \mathbf{W}_j = L^2(\mathbb{R}, \mathbb{C}[r])$. Since $\Psi(t) \in \mathbf{W}_0 \subseteq \mathbf{V}_1$, then $\Psi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} \mathbf{H}_k \Phi(2t-k)$. Combining this formula with (3.1) gives us

$$\sum_{k\in\mathbb{Z}} \mathbf{G}_k \mathbf{H}^{\dagger}_{2l+k} = 0_r, \quad l \in \mathbb{Z}. \tag{3.4}$$

Now if $\widehat{\mathbf{H}}(f) = \sum_{k\in\mathbb{Z}} \mathbf{H}_k e^{-2\pi i k f}$ then

$$\widehat{\mathbf{H}}(f)\widehat{\mathbf{G}}^{\dagger}(f) + \widehat{\mathbf{H}}(f+\frac{1}{2})\widehat{\mathbf{G}}^{\dagger}(f+\frac{1}{2}) = 0_r, \tag{3.5}$$

and $\widehat{\Psi}(f) = \frac{1}{\sqrt{2}}\widehat{H}(\frac{f}{2})\widehat{\phi}(\frac{f}{2})$. If $\{\Psi(t-k) : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{W}_0$ then

$$\langle \Psi, \Psi(t-k)\rangle = \int_{\mathbb{R}} \Psi(t)\Psi(t-k)dt = \mathbf{I}_r\delta_{k0} \quad k \in \mathbb{Z},$$

which implies the following relation for the matrix of wavelet mask function:

$$\sum_{k\in\mathbb{Z}} \mathbf{H}_k \mathbf{H}^{\dagger}_{2l+k} = I_r \delta_{l0}, \quad l \in \mathbb{Z}. \tag{3.6}$$

This is equivalent to

$$\widehat{\mathbf{H}}(f)\widehat{\mathbf{H}}^{\dagger}(f) + \widehat{\mathbf{H}}(f+\frac{1}{2})\widehat{\mathbf{H}}^{\dagger}(f+\frac{1}{2}) = 2I_r. \tag{3.7}$$

Define $\widehat{\mathbf{M}}(f) = \begin{pmatrix} \widehat{\mathbf{G}}(f) & \widehat{\mathbf{G}}(f+\frac{1}{2}) \\ \widehat{\mathbf{H}}(f) & \widehat{\mathbf{H}}(f+\frac{1}{2}) \end{pmatrix}$ then equations (3.3),(3.5),(3.7) all together are equivalent to

$$\widehat{\mathbf{M}}(f)\widehat{\mathbf{M}}^{\dagger}(f) = 2I_{2r}, \tag{3.8}$$

which means $\widehat{\mathbf{M}}(f)$ is a paraunitary matrix.

3.2. **Construction of filters.** After constructing the mask function representation, now we are ready to describe and build filters. Suppose that $\widehat{\mathbf{G}}(f)$ is a finite polynomial matrix in $e^{-2\pi i f}$, i.e. can be written in the form $\widehat{\mathbf{G}}(f) = \sum_{l=0}^{L'-1} \mathbf{G}_l e^{-2\pi i f l}$ with $\widehat{\mathbf{G}}(0) = \sqrt{2}\mathbf{I}_r$, and satisfies (3.1). Then from [8] if

$$\inf_{|f| \leq \frac{1}{4}} |\lambda_l[\widehat{\mathbf{G}(f)}]| > 0 \tag{3.9}$$

for any eigenfunction $\lambda_l[\widehat{\mathbf{G}}(f)]$ of polynomial matrix $\widehat{\mathbf{G}}(f)$, the solution $\Phi(t)$ of the two-scale dilation equation is a matrix-valued scaling function for a matrix-valued MRA, and $\{\Psi_{j,k}(t) = 2^{\frac{j}{2}}\Psi(2^j t - k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for matrix-valued space $L^2(\mathbb{R}, \mathbb{C}[r])$. For designing the matrix filters with transforms $\widehat{\mathbf{G}}(f)$ and $\widehat{\mathbf{H}}(f)$ that satisfies (3.2) and for that $\widehat{\mathbf{M}}(f)$ is paraunitary , we consider

$$\widehat{\mathbf{G}}(f) = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(\mathbf{I}_r + e^{\epsilon 2\pi i f}\widehat{\mathbf{P}}(2f)), \quad \epsilon \in \{-1, 1\} \tag{3.10}$$

where γ is a finite integer and $\widehat{\mathbf{P}}(2f)$ is a (normalized) paraunitary matrix, i.e. $\widehat{\mathbf{P}}(f)\widehat{\mathbf{P}}^{\dagger}(f) = \mathbf{I}_r$ which satisfies $\widehat{\mathbf{P}}(f+1) = \widehat{\mathbf{P}}(f)$, and such that $\widehat{\mathbf{P}}(0) = \mathbf{I}_r$. The matrix $\widehat{\mathbf{G}}(f)$ satisfies conditions (3.1) and (3.2). Notice that the eigenvalues of the polynomial matrix $\widehat{\mathbf{G}}(f)$ are related to the eigenvalues of $\widehat{\mathbf{P}}(2f)$ via $\lambda_l[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}\{1 + e^{\epsilon 2\pi i f}\lambda_l[\widehat{\mathbf{P}}(2f)]\}$. Since $\widehat{\mathbf{M}}(f)$ is paraunitary, $\widehat{\mathbf{H}}(f)$ may be chosen as

$$\widehat{\mathbf{H}}(f) = e^{-2\pi i f(L'-1+\delta)}\widehat{\mathbf{G}}^{\dagger}(f + \frac{1}{2}) \tag{3.11}$$

where L' is the design length of the filter $\mathbf{G}_l$, and $\delta \in \{0, 1\}$ is chosen so that $L' - 1 + \delta$ is odd, because by 3.5

$$\begin{aligned} &\widehat{\mathbf{H}}(f)\widehat{\mathbf{G}}^{\dagger}(f) + \widehat{\mathbf{H}}(f + \frac{1}{2})\widehat{\mathbf{G}}^{\dagger}(f + \frac{1}{2}) \\ &= e^{-2\pi i f(L'-1+\delta)}[\widehat{\mathbf{G}}^{\dagger}(f + \frac{1}{2})\widehat{\mathbf{G}}^{\dagger}(f) + e^{-\pi i(L'-1+\delta)}\widehat{\mathbf{G}}^{\dagger}(f)\widehat{\mathbf{G}}^{\dagger}(f + \frac{1}{2})] \\ &= e^{-2\pi i f(L'-1+\delta)}[\widehat{\mathbf{G}}^{\dagger}(f + \frac{1}{2})\widehat{\mathbf{G}}^{\dagger}(f) - \widehat{\mathbf{G}}^{\dagger}(f)\widehat{\mathbf{G}}^{\dagger}(f + \frac{1}{2})] = 0_r, \end{aligned}$$

which provide $\widehat{\mathbf{G}}(f)$ is commutative in the sense that $\widehat{\mathbf{G}}(f)\widehat{\mathbf{G}}(f + \frac{1}{2}) = \widehat{\mathbf{G}}(f + \frac{1}{2})\widehat{\mathbf{G}}(f)$, and indeed this condition holds when $\widehat{\mathbf{G}}(f)$ is defined as in (3.10). The matrix $\widehat{\mathbf{H}}$ given by (3.11) is a polynomial which can be written in the form

$$\widehat{\mathbf{H}} = \sum_{m=\delta}^{L'-1+\delta} (-1)^{L'-1+\delta-m}\mathbf{G}^{\dagger}_{L'-1+\delta-m}e^{-2\pi i f m}.$$

If L' is even (and $\delta = 0$), then comparison with $\widehat{\mathbf{H}} = \sum_{l=0}^{L'-1}\mathbf{H}_l e^{-2\pi i f l}$ we obtain $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}^{\dagger}_{L'-l-1}$ for $l = 0, 1, \ldots, L' - 1$ and we set $L = L'$. If L' is odd ($\delta = 1$) we can increase the filter length to an even length $L' + 1$ by setting $\mathbf{G}_{L'} = 0_r$. Then we have $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}^{\dagger}_{(L'+1)-l-1}$ for $l =$

$0,\ldots,L'$, with $\mathbf{H}_0 = 0_r$. In this case we set $L = L' + 1$. For constructing the matrix $\widehat{\mathbf{P}}(f)$ we first consider the class of paraunitary matrices, defined by $\widehat{\mathbf{P}}(f) = \widehat{\mathbf{U}}(f)\widehat{\mathbf{D}}(f)\mathbf{U}^{\dagger}(f)$, where $\widehat{\mathbf{U}}(f)$ is an arbitrary (normalized) paraunitary polynomial matrix with $\widehat{\mathbf{U}}(0) = \mathbf{I}_r$, and $\widehat{\mathbf{D}}(f)$ is a diagonal matrix with diagonal elements $\widehat{\mathbf{D}}_{l,l} = e^{-2\pi i f k_l}, k_l \in \{0,1\}$. Using the general lattice structure, the $r\times r$-matrix $\widehat{\mathbf{U}}(f)$ may be constructed by $\widehat{\mathbf{U}}(f) = \widehat{\mathbf{U}}_q(f),\ldots,\widehat{\mathbf{U}}_1(f)\mathbf{F}$, where q is a positive integer, $\mathbf{F}$ is an $r\times r$ constant unitary matrix, i.e. $\mathbf{F}^{\dagger}\mathbf{F} = \mathbf{F}\mathbf{F}^{\dagger}$, and $\widehat{\mathbf{U}}_l(f) = I_r + (e^{2\pi i f} - 1)\mathbf{z}_l\mathbf{z}_l^{\dagger}\ \ l = 0,\ldots,q$ with $\widehat{\mathbf{z}}_l^{\dagger}\mathbf{z}_l = 1$, unit-norm constant $r\times 1$-vectors. The advantage of this construction is that the matrices $\widehat{\mathbf{D}}(f)$ and $\widehat{\mathbf{P}}(f)$ are similar and hence have the same eigenvalues, and those of $\widehat{\mathbf{D}}(f)$ are known. It is thus possible to compute the eigenvalues of $\widehat{\mathbf{G}}(f)$ to check that the sufficient condition (3.9) is satisfied.

4. Main results for $\mathbb{C}l(\mathbb{R}^4)$)-MRA

Case I:
Let $r = 4$, by the previous section $\widehat{\mathbf{D}}_{l,l} = e^{-2\pi i k f},\ k \in \{0,1\}, l = 1,2,3,4$. So we have

$$\widehat{\mathbf{P}}(f) = \widehat{\mathbf{U}}(f)\widehat{\mathbf{D}}(f)\mathbf{U}^{\dagger}(f)$$

If $\widehat{\mathbf{U}}(f) = \mathbf{I}_4$, $\widehat{\mathbf{U}}$ is a paraunitary polynomial matrix which $\widehat{\mathbf{U}}(0) = \mathbf{I}_4$, so $\widehat{\mathbf{P}}(f) = e^{-2\pi i k f}\mathbf{I}_4$, this gives the diagonal matrix $\widehat{\mathbf{G}}(f) = \frac{e^{2\pi i f\gamma}}{\sqrt{2}}(1+e^{(\epsilon-2k)2\pi i f})\mathbf{I}_4$. $\widehat{\mathbf{G}}(f)$ has only one eigenvalue which is repeated and is $\lambda[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f\gamma}}{\sqrt{2}}(1 + e^{(\epsilon-2k)2\pi i f})$. Now if we set $\epsilon = 1$ we obtain

$$\lambda[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f\gamma}}{\sqrt{2}}(1 + e^{2\pi i f}), (k = 0)$$

$$\lambda[\widehat{\mathbf{G}}(f)] = \frac{e^{2\pi i f\gamma}}{\sqrt{2}}(1 + e^{-2\pi i f}), (k = 1)$$

which in both case the condition $|\lambda[\widehat{\mathbf{G}}(f)]| = \sqrt{1+\cos 2\pi f} > 0\ \ ,\text{for}|f| \leq \frac{1}{4}$, is fullfaith. Hence the sufficient condition (3.9) is satisfied.
If we set $\gamma = 0, \epsilon = 1, k = 1$, then

$$\widehat{\mathbf{G}}(f) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+e^{-2\pi i f} & 0 & 0 & 0 \\ 0 & 1+e^{-2\pi i f} & 0 & 0 \\ 0 & 0 & 1+e^{-2\pi i f} & 0 \\ 0 & 0 & 0 & 1+e^{-2\pi i f} \end{pmatrix}.$$

Let $f = 0$, then $\widehat{\mathbf{G}}(0) = \sqrt{2}\mathbf{I}_4$, $\widehat{\mathbf{G}}(\frac{1}{2}) = 0_4$ and in comparison with $\widehat{\mathbf{G}}(f) = \sum_{l=0}^{L'-1} \mathbf{G}_l e^{-2\pi i f l}$ we have

$$\widehat{\mathbf{G}}(f) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} e^{-2\pi i f}.$$

This means that $\mathbf{G}_0 = \mathbf{G}_1 = \frac{1}{\sqrt{2}}\mathbf{I}_4$ so, $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}^{\dagger}_{L-l-1}$ for $l = 0, 1$.

Case II:

From now on we consider $\widehat{\mathbf{G}}(f) = \frac{e^{2\pi i f \gamma}}{\sqrt{2}}(\mathbf{I}_4 + e^{\epsilon 2\pi i f}\widehat{\mathbf{P}}(2f))$, we can make $\widehat{\mathbf{P}}(f)$ as

$$\widehat{\mathbf{P}}(f) = \widehat{\mathbf{U}}(f)\widehat{\mathbf{D}}(f)\mathbf{U}^{\dagger}(f)$$

(for $L^2_{\mathbf{M}_Q}(\mathbb{R}, \mathbb{C}[4])$ we set $\widehat{\mathbf{U}}(f) \in \mathbf{M}_Q \bigcap \mathbf{U}(4)$).
Set $q = 1$ and $\mathbf{F} = 4 \times 4$-rotation matrix

$$\mathbf{F} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix},$$

(note that $\mathbf{F} \in \mathbf{M}_Q$). Then $\widehat{\mathbf{U}}(f) = \widehat{\mathbf{U}_1}(f)\mathbf{F}$ such that $\widehat{\mathbf{U}_1}(f) = \mathbf{I}_4 + (e^{2\pi i f} - 1)\mathbf{z}_1\mathbf{z}_1^{\dagger}$.
Now let $\mathbf{z}_1 = \frac{e^{i\theta}}{\alpha}(a, b, c, d)^T$ so $\mathbf{z}_1^{\dagger} = \frac{e^{-i\theta}}{\alpha}(a, b, c, d)$ such that $\alpha = a^2+b^2+c^2+d^2$.
For instant if $(a, b, c, d) = (0, 0, 0, \alpha), \alpha \in \mathbb{R}$, then $\mathbf{z}_1\mathbf{z}_1^{\dagger}$ is a 4×4-matrix with all entiers zero except $e_{4,4} = 1$, so $\mathbf{U}_1(f)$ is the same matrix with $e_{4,4} = e^{2\pi i f}$ and by choosing $\mathbf{D}$ such that $\mathbf{D}_{1,1} = 1, \mathbf{D}_{2,2} = \mathbf{D}_{3,3} = \mathbf{D}_{4,4} = e^{-2\pi i f}$ finally we have:

(4.1)
$$\widehat{\mathbf{G}}(f) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2\theta + e^{-2\pi i f} + e^{-4\pi i f}\sin^2\theta & \sin\theta\cos\theta - e^{-4\pi i f}\sin\theta\cos\theta & 0 & 0 \\ \sin\theta\cos\theta - e^{-4\pi i f}\sin\theta\cos\theta & e^{-2\pi i f} + \sin^2\theta + e^{-4\pi i f}\cos_2\theta & 0 & 0 \\ 0 & 0 & 2e^{-2\pi i f} & 0 \\ 0 & 0 & 0 & 2e^{-2\pi i f} \end{pmatrix}.$$

This means that $\mathbf{G}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta & 0 & 0 \\ \sin\theta\cos\theta & \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\mathbf{G}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$\mathbf{G}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin^2\theta & \sin\theta\cos\theta & 0 & 0 \\ \sin\theta\cos\theta & \cos^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and since $L' - 1 = 3$ then $L' = 4$ so $\delta = 0$.
Then we set $L = L' = 4$.
Now by $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}^{\dagger}_{L'-l-1}, \quad (l = 0, 1, 2, 3)$ we have

$$\mathbf{H}_0 = -\mathbf{G}_3^{\dagger} = 0_4,\ \mathbf{H}_1 = \mathbf{G}_2^{\dagger},\ \mathbf{H}_2 = -\mathbf{G}_1^{\dagger},\ \mathbf{H}_3 = \mathbf{G}_0^{\dagger}.$$

So from (3.1) and (3.2) we obtain the desired wavelets.

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