

## New Integral Inequalities through the $\varphi$ -Preinvexity

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ABSTRACT. In this note, we give some estimates of the generalized quadrature formula of Gauss-Jacobi type for  $\varphi$ -preinvex functions.

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### 1. INTRODUCTION

It is well known that the convexity plays an important and very central role in many areas, such as economics, finances, optimization, and games theory. Due to its diverse applications this concept has been extended and generalized in several directions.

We recall that the function  $f : I \rightarrow \mathbb{R}$  where  $I$  is an interval of  $\mathbb{R}$ , is said to be  $\varphi$ -convex, if the following inequality

$$f(tx + (1-t)y) \leq f(y) + t\varphi(f(y), f(x))$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$  such that  $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a bifunction (see [3]).

The generalized quadrature formula of Gauss-Jacobi type has the following form

$$\int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^m B_{m,k} f(\gamma_k) + \mathfrak{R}_m[f], \quad (1.1)$$

where  $B_{m,k}$  are the Christoffel coefficients,  $\gamma_k$  are the roots of the Jacobi polynomial of degree  $m$ , and  $\mathfrak{R}_m[f]$  is the remainder term (see[12]).

In [10], Özdemir et al. gave the estimate of the left hand sides of equality (1.1) when the function  $f$  is quasi-convex on  $[a, b] \subset \mathbb{R}$  with  $0 \leq a < b < \infty$ , as follows

$$\int_a^b (x-a)^p (b-x)^q f(x) dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) \times \max\{f(a), f(b)\}.$$

Ahmad [1], gave the the estimates of the left hand sides of equality (1.1) when  $|f|$ , and  $|f|^l$  are  $P$ -preinvex, and prequasiinvex functions. In [5, 6], Liu discussed the cases where certain power of the modulus of the function  $f$  is quasi-convex, and  $(\alpha, m)$ -convex, and  $P$ -convex functions. İşcan et al. [4], treated the equality (1.1) in the cases where  $f$  and  $|f|^\lambda$  are harmonically convex functions. In [8], Muddassar et al. discussed the equality (1.1) in the cases where  $|f|$ ,  $|f|^{\frac{k}{k-1}}$  and  $|f|^l$  are  $s$ - $(\alpha, m)$ -convex functions. In [9], Noor et al. established the estimates of left hand side of (1.1) for strongly generalized harmonic convex function with modulus  $c > 0$ . About some books dealing with different types of inequalities see for instance [2, 7, 11].

Motivated by the above results, in the present note we give the estimates of the left hand side of (1.1) for  $\varphi$ -convex functions.

## 2. MAIN RESULTS

In what follows, we assume that the interval  $[a, b] \subset \mathbb{R}^+$ ,  $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a bifunction, and  $p, q > 0$ .

For establishing our results we need the following lemma

**Lemma 2.1.** [8] *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  such that  $f \in L([a, b])$ ,  $a < b$ . Then the equality*

$$\int_a^b (x-a)^p (b-x)^q f(x) dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta + (1-t)b) dt$$

*holds for some fixed  $p, q > 0$ .*

**Theorem 2.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an continuous and integrable function on  $[a, b]$ . If  $f$  is  $\varphi$ -convex function, we have*

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x) dx \\ & \leq (b-a)^{p+q+1} \beta(q+1, p+1) \left( |f(b)| + \frac{q+1}{p+q+2} \varphi(|f(b)|, |f(a)|) \right), \end{aligned} \quad (2.1)$$

where  $\beta(.,.)$  is the Beta function.

*Proof.* From Lemma 2.1, modulus and  $\varphi$ -convexity of  $f$ , we have

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x) dx \\ & \leq \int_a^b (x-a)^p (b-x)^q |f(x)| dx \\ & \leq (b-a)^{p+q+1} |f(b)| \int_0^1 (1-t)^p t^q dt \\ & \quad + (b-a)^{p+q+1} \varphi(|f(b)|, |f(a)|) \int_0^1 (1-t)^p t^{q+1} dt \\ & = (b-a)^{p+q+1} |f(b)| \beta(q+1, p+1) \\ & \quad + (b-a)^{p+q+1} \varphi(|f(b)|, |f(a)|) \beta(q+2, p+1) \\ & = (b-a)^{p+q+1} \beta(q+1, p+1) \left( |f(b)| + \frac{q+1}{p+q+2} \varphi(|f(b)|, |f(a)|) \right), \end{aligned}$$

which is the desired result.  $\square$

**Theorem 2.3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and integrable function on  $[a, b]$ , and let  $\lambda > 1$ . If  $|f|^\lambda$  is  $\varphi$ -convex function, we have*

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x) dx \leq (b-a)^{p+q+1} (\beta(q+1, p+1)) \\ & \quad \times \left( |f(b)|^\lambda + \frac{q+1}{p+q+2} \left| \varphi(|f(b)|^\lambda, |f(a)|^\lambda) \right| \right)^{\frac{1}{\lambda}}. \end{aligned} \quad (2.2)$$

*Proof.* From Lemma 2.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
& \int_a^b (x-a)^p (b-x)^q f(x) dx \\
& \leq (b-a)^{p+q+1} \left( \int_0^1 (1-t)^p t^q dt \right)^{1-\frac{1}{\lambda}} \left( \int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|^\lambda dt \right)^{\frac{1}{\lambda}} \\
& = (b-a)^{p+q+1} (\beta(q+1, p+1))^{1-\frac{1}{\lambda}} \left( \int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|^\lambda dt \right)^{\frac{1}{\lambda}}.
\end{aligned}$$

Since  $|f|^\lambda$  is  $\varphi$ -convex, we deduce

$$\begin{aligned}
& \int_a^b (x-a)^p (b-x)^q f(x) dx \\
& \leq (b-a)^{p+q+1} (\beta(q+1, p+1))^{1-\frac{1}{\lambda}} \\
& \quad \times \left( \int_0^1 (1-t)^p t^q |f(b)|^\lambda dt + \int_0^1 (1-t)^p t^{q+1} \left| \varphi(|f(b)|^\lambda, |f(a)|^\lambda) \right| dt \right)^{\frac{1}{\lambda}} \\
& = (b-a)^{p+q+1} (\beta(q+1, p+1)) \\
& \quad \times \left( |f(b)|^\lambda + \frac{q+1}{p+q+2} \left| \varphi(|f(b)|^\lambda, |f(a)|^\lambda) \right| \right)^{\frac{1}{\lambda}},
\end{aligned}$$

which is the desired result.  $\square$

**Theorem 2.4.** *Suppose that all the assumptions of Theorem 2.3 are satisfied, then we have*

$$\begin{aligned}
& \int_a^b (x-a)^p (b-x)^q f(x) dx \leq \frac{(b-a)^{p+q+1}}{2^{\frac{1}{\lambda}}} \left( \beta \left( \frac{q\lambda}{\lambda-1} + 1, \frac{p\lambda}{\lambda-1} + 1 \right) \right)^{1-\frac{1}{\lambda}} \\
& \quad \times \left( 2|f(b)|^\lambda + \left| \varphi(|f(b)|^\lambda, |f(a)|^\lambda) \right| \right)^{\frac{1}{\lambda}}. \tag{2.3}
\end{aligned}$$

*Proof.* From Lemma 2.1, properties of modulus, and Hölder inequality, we have

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x) dx \\ & \leq (b-a)^{p+q+1} \left( \int_0^1 (1-t)^{\frac{p\lambda}{\lambda-1}} t^{\frac{q\lambda}{\lambda-1}} dt \right)^{1-\frac{1}{\lambda}} \left( \int_0^1 |f(ta+(1-t)b)|^\lambda dt \right)^{\frac{1}{\lambda}} \\ & = (b-a)^{p+q+1} \left( \beta \left( \frac{q\lambda}{\lambda-1} + 1, \frac{p\lambda}{\lambda-1} + 1 \right) \right)^{1-\frac{1}{\lambda}} \left( \int_0^1 |f(ta+(1-t)b)|^\lambda dt \right)^{\frac{1}{\lambda}}. \end{aligned}$$

Since  $|f|^\lambda$   $\varphi$ -convex, we get

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x) dx \\ & \leq (b-a)^{p+q+1} \left( \beta \left( \frac{q\lambda}{\lambda-1} + 1, \frac{p\lambda}{\lambda-1} + 1 \right) \right)^{1-\frac{1}{\lambda}} \\ & \quad \times \left( |f(b)|^\lambda \int_0^1 dt + \left| \varphi \left( |f(b)|^\lambda, |f(a)|^\lambda \right) \right| \int_0^1 t dt \right)^{\frac{1}{\lambda}} \\ & = \frac{(b-a)^{p+q+1}}{2^{\frac{1}{\lambda}}} \left( \beta \left( \frac{q\lambda}{\lambda-1} + 1, \frac{p\lambda}{\lambda-1} + 1 \right) \right)^{1-\frac{1}{\lambda}} \left( 2|f(b)|^\lambda + \left| \varphi \left( |f(b)|^\lambda, |f(a)|^\lambda \right) \right| \right)^{\frac{1}{\lambda}}, \end{aligned}$$

which is the desired result.  $\square$

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