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Linear Functions Preserving Multivariate and Directional Majorization

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Abstract. Let V and W be two real vector spaces and let $\sim$ be a relation on both V and W. A linear function $T : V \to W$ is said to be a linear preserver (respectively strong linear preserver) of $\sim$ if $Tx \sim Ty$ whenever $x \sim y$ (respectively $Tx \sim Ty$ if and only if $x \sim y$). In this paper we characterize all linear functions $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ which preserve or strongly preserve multivariate and directional majorization.

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1. Introduction

Let $\mathbf{M}_{n,m}$ be the vector space of all real $n \times m$ matrices. An $n \times n$ matrix $D = [d_{ij}]$ is called doubly stochastic provided that the entries of D are all nonnegative and $\sum_{k=1}^{n} d_{ik} = \sum_{k=1}^{n} d_{kj} = 1$ for every $i, j \in \{1, \cdots, n\}$. Let X and Y belong to $\mathbf{M}_{n,m}$, we say X is multivariate majorized by Y (written $X \prec_m Y$) if $X = DY$ for some $n \times n$ doubly stochastic matrix D. A generalized concept of multivariate majorization was introduced in [3]. For X and Y belong

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to $\mathbf{M}_{n,m}$, it is said that X is directional majorized by Y (written $X \prec_d Y$) if for every $a \in \mathbb{R}^m$ there exists a doubly stochastic matrix D_a such that $Xa = D_a Ya$. When $m = 1$, the definition of multivariate majorization and directional majorization reduce to the classical concept of vector majorization. Vector majorization is a much studied concept in linear algebra and its applications, for more details about vector majorization see [1] and [5]. Some of our notations are explained next.

$\mathcal{P}_n$; The set of all $n \times n$ permutation matrices.

$\mathbf{J}$; The $n \times n$ matrix with all entries equal to 1.

$X = [x_1|\cdots|x_m]$; An $n \times m$ matrix with $x_j \in \mathbb{R}^n$ as the j^{th} column of X.

trx ; The summation of all components of a vector $x \in \mathbb{R}^n$.

About linear functions preserving multivariate and directional majorization on $\mathbf{M}_{n,m}$, Li and Poon obtained the following interesting result in [4].

Proposition 1.1. *Let T be a linear operator on $\boldsymbol{M}_{n,m}$. Then T preserves multivariate majorization if and only if T preserves directional majorization if and only if one of the following holds:*

(a) There exist $A_1, \cdots, A_m \in \boldsymbol{M}_{n,m}$ such that $T(X) = \sum_{j=1}^{m}(trx_j)A_j$.

(b) There exist $R, S \in M_m$ and $P \in \mathcal{P}_n$ such that $T(X) = PXR + \boldsymbol{J}XS$.

The above proposition is in fact a generalization of the following proposition which has been proved by Ando in [2].

Proposition 1.2. *A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves vector majorization if and only if one of the following holds:*

(i) $Tx = (trx)a$ for some $a \in \mathbb{R}^n$.

(ii) $Tx = \alpha Px + \beta(trx)e = \alpha Px + \beta \boldsymbol{J}x$ for some $\alpha, \beta \in \mathbb{R}$ and $P \in \mathcal{P}_n$.

Our main result is a generalization of Proposition 1.1. In fact, we prove the following theorem.

Theorem 1.3. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,k}$ be a linear function. Then T preserves multivariate majorization if and only if T preserves directional majorization if and only if one of the following holds:*

(a) There exist $A_1, \cdots, A_m \in \boldsymbol{M}_{n,k}$ such that $TX = \sum_{i=1}^{m}(trx_i)A_i$.

(b) There exist $P \in \mathcal{P}_n$ and $R, S \in \boldsymbol{M}_{m,k}$ such that $TX = PXR + \boldsymbol{J}XS$.

2. Main result

We state the following statements to prove the main theorem. The following proposition is proved in [4].

Proposition 2.1. *Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two linear preservers of vector majorization which satisfy :*

$$T_1Qy + \gamma T_2Qy \prec T_1y + \gamma T_2y \quad \forall\gamma \in \mathbb{R},\ \forall y \in \mathbb{R}^n,\ \forall Q \in \mathcal{P}_n. \tag{2.1}$$

Then T_1, T_2 are either of the form (i) or (ii) in Proposition 1.2 with the same P.

First, we prove a special case of Theorem 1.3, in which $k = 1$.

Lemma 2.2. *A linear function $T : \boldsymbol{M}_{n,m} \to \mathbb{R}^n$ preserves multivariate majorization if and only if one of the following holds:*
(i) *There exist $a_1, \cdots, a_m \in \mathbb{R}^n$ such that $TX = \sum_{j=1}^m (trx_j)a_j$.*
(ii) *There exist $a, b \in \mathbb{R}^m$ and $P \in \mathcal{P}_n$ such that $TX = PXa + \boldsymbol{J}Xb$.*

Proof. Define $T' : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by $T'X = [TX|0]$ where 0 denotes the $n \times (m-1)$ zero matrix. Clearly T' is a linear operator which preserves multivariate majorization. Then by Proposition [4], T' has one of the following forms;
(a) $T'(X) = \sum_{i=1}^m (trx_i)B_i$ for some $B_1, \cdots, B_m \in \mathbf{M}_{n,m}$. So $TX = \sum_{j=1}^m (trx_j)a_j$, where a_j is the first column of B_j for any j $(1 \le j \le n)$ and hence (i) holds.
(b) $T'(X) = PXR' + JXS'$ for some $P \in \mathcal{P}_n$ and some $R', S' \in \mathbf{M}_m$. So $TX = PXa + \mathbf{J}Xb$ where a and b are the first columns of R and S respectively, and hence (ii) holds. □

Lemma 2.3. *Let $T_1, T_2 : \boldsymbol{M}_{n,m} \to \mathbb{R}^n$ be two linear functions. If $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,2}$ defined by $TX = [T_1X|T_2X]$ preserves multivariate majorization, then T_1, T_2 are both either of the form (i) or (ii) in Lemma 2.2 with the same P.*

Proof. If $m = 1$ then T_1, T_2 satisfy the conditions of Proposition 2.1 and hence T_1, T_2 are either of the form (i) or (ii) in Proposition 1.2 with the same P. If $m \ge 2$, define $T' : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ by $T'(X) = [TX|0]$ where 0 denotes the $n \times (m-2)$ zero matrix. Clearly T' is an operator which preserves multivariate majorization. Therefore by Proposition 1.1, either $T'(X) = \sum_{i=1}^m (trx_i)B_i$, for some $B_1, \cdots, B_m \in \mathbf{M}_{n,m}$ and hence $T(X) = \sum_{i=1}^m (trx_i)A_i$ where $B_i = [A_i|*]$ and $*$ is an $n \times (m-2)$ block for every i $(1 \le i \le m)$, or $T'(X) = PXR' + JXS'$ for some $P \in P_n, R', S' \in \mathbf{M}_m$ and hence $T(X) = PXR + JXS$ where $R' = [R|*_1]$, $S' = [S|*_2]$ and $*_1, *_2$ are two $n \times (m-2)$ blocks. □

Proof of Theorem 1.3. If T satisfies (a) or (b), trivially T preserves multivariate and directional majorization. Conversely, let T be a linear preserver of multivariate majorization. Then there exist linear functions $T_i : \mathbf{M}_{n,m} \to \mathbb{R}^n$, $i = 1, \cdots, k$ such that $TX = [T_1X| \cdots |T_kX]$. It is easy to see that T_i preserves multivariate majorization for every i $(1 \le i \le k)$. By Lemma 2.2 and Lemma 2.3, either every T_i satisfies condition (i) or (ii) of Lemma 2.2. If for every i $(1 \le i \le k)$ $T_i(X) = \sum_{j=1}^m (trx_j)a_j^i$ for some $a_1^i, \cdots, a_m^i \in \mathbb{R}^n$, then $T(X) = [\sum_{j=1}^m (trx_j)a_j^1 | \sum_{j=1}^m (trx_j)a_j^2 | \cdots | \sum_{j=1}^m (trx_j)a_j^m] = \sum_{j=1}^m (trx_j)A_j$, for some $A_j \in \mathbf{M}_{n,k}$. Hence T satisfies condition (i). If for every i $(1 \le i \le k)$, $T_i(X) = PXa_i + JXb_i$ for some $a_i, b_i \in \mathbb{R}^m$ and $P \in \mathcal{P}_n$. Then $TX = [PXa_i + JXb_i| \cdots |PXa_k + JXb_k] = PX[a_1| \cdots |a_k] + JX[b_1| \cdots |b_k] = PXR + JXS$ for some $R, S \in \mathbf{M}_{m,k}$. Thus T satisfies condition (ii). If T preserves directional majorization it is easy to see that the following condition holds:

$$TX \prec_d TY \qquad \text{whenever} \qquad X \prec_m Y. \tag{2.2}$$

Now, if one replace multivariate majorization preserving by condition (2.2) in the previous lemmas, all proofs are valid. Then T satisfies conditions (a) or (b). □

Now, we state the following lemma to characterize all strong linear preserver of multivariate and directional majorization from $\mathbf{M}_{n,m}$ to $\mathbf{M}_{n,k}$

Lemma 2.4. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,k}$ be a linear function of the form $T(X) = DXR + JXS$, for some $R, S \in \mathbf{M}_{m,k}$ and invertible doubly stochastic $D \in \mathbf{M}_n$. Then T is injective if and only if R and $(R + nS)$ are full-rank matrices.*

Proof. Without loss of generality we may assume that $D = I$. Since $dim(KerT) + rank(T) = nm$, if $k < m$ then $dim(KerT) \geq 1$. Therefore T is not injective. If $k \geq m$, the matrix representation of T with respect to the standard bases of $\mathbf{M}_{n,m}$ and $\mathbf{M}_{n,k}$ is similar to the following block matrix:

$$\begin{pmatrix} R+nS & & & * \\ & R & & \\ & & \ddots & \\ 0 & & & R \end{pmatrix} \in \mathbf{M}_{nk,nm} \quad . \tag{2.3}$$

Therefore T is injective if and only if R and $(R+nS)$ are full-rank matrices. □

Theorem 2.5. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,k}$ be a linear function. Then T strongly preserves multivariate majorization if and only if T strongly preserves directional majorization if and only if there exist $P \in \mathcal{P}_n$ and $R, S \in M_{m,k}$ such that R , $(R + nS)$ are full-rank matrices and $TX = PXR + \boldsymbol{J}XS$.*

Proof. It is clear that every strong linear preserver of multivariate majorization is injective. So by Theorem 1.3 and Lemma 2.4, $TX = PXR + \mathbf{J}XS$ for some $R, S \in M_{m,k}$ such that R , $(R + nS)$ are full-rank matrices. The other side is trivial. □

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