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The Study of Some Boundary Value Problems Including Fractional Partial Differential Equations with non-Local Boundary Conditions

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Abstract. In this paper, we consider some boundary value problems (BVP) for fractional order partial differential equations (FPDE) with non-local boundary conditions. The solutions of these problems are presented as series solutions analytically via modified Mittag-Leffler functions. These functions have been modified by authors such that their derivatives are invariant with respect to fractional derivative. The presented solutions for these problems are as infinite series. Convergence of series solutions and uniqueness of them are established by general theory of mathematical analysis and theory of ODEs.

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1. Introduction

Fractional calculus and fractional differential equations have improved more in recent years [16,17]. Moreover, regarding the integral representation of the solutions of fractional differential equations, these solutions are not very computational, because their integral represnetations have complex integrands, especially include terms of Gamma function [3,11].
In this paper, we will search the form of solutions for FPDE as series solutions by modified Mittag-Leffler functions. These functions are invariant with respect to fractional order as same as Euler exponential function to ordinary derivative. The solutions are presented in this paper are based on Mittag-Lefller functions which are very computational [1,2]. One can constructs an approxmiate solution by choosing specific number of terms from series solution. The general modified Mittag- Leffler function can be written in the following form [11,12,13]:

$$h_p(x,r) = \sum_{k=1}^{\infty} \frac{r^k x^{kp-1}}{(kp-1)!}, \tag{1.1}$$

where the real number p is the fractional step derivative and r is and unknown parameter. It is easy to see that, the following relation is valid

$$D^{np} h_p(x,r) = r^n . h_p(x,r), \tag{1.2}$$

similar to exponential function $D^n e^{rx} = r^n e^{rx}$. The parameter r is usually root of characteristic equation. For example we consider the following fractional ordinary differential equation:

$$y^{(\frac{2}{3})}(x) - 3y^{(\frac{1}{3})}(x) + 2y(x) = 0, \tag{1.3}$$

note that $p = \frac{1}{3}$, $np = \frac{2}{3}$ and $n = 2$, we consider the following relations:

$$D^{\frac{1}{3}} h_{\frac{1}{3}}(x,r) = r h_{\frac{1}{3}}(x,r), \quad D^{\frac{2}{3}} h_{\frac{1}{3}}(x,r) = r^2 h_{\frac{1}{3}}(x,r). \tag{1.4}$$

After replacing the above amounts in the fractional equation (2) we have:

$$r^2 h_{\frac{1}{3}}(x,r) - 3r h_{\frac{1}{3}}(x,r) + 2h_{\frac{1}{3}}(x,r) = 0, \tag{1.5}$$

and characteristic equation will be $r^2 - 3r + 2 = 0$. The roots of this equation are $r_1 = 1, r_2 = 2$. Then the general solution of equation (2) is as follows:

$$y(x) = c_1 h_{\frac{1}{3}}(x,1) + c_2 h_{\frac{1}{3}}(x,2) \tag{1.6}$$

$$= c_1 \sum_{k=1}^{\infty} \frac{(1)^k x^{(\frac{k}{3})-1}}{((\frac{k}{3})-1)!} + c_2 \sum_{k=1}^{\infty} \frac{(2)^k x^{(\frac{k}{3})-1}}{((\frac{k}{3})-1)!}. \tag{1.7}$$

2. BVP for FPDE on Unbounded Domain

In this section, we consider some BVP for FPDE in unbounded domain. For this, consider the following problem in the first quarter:

$$D_x^\alpha u(x,y) = D_y^\alpha u(x,y), \quad x>0, y>0, \tag{2.1}$$

with non-local boundary condition

$$u(t,0) = au(0,t) + \varphi(t), \quad t \geqslant 0, \tag{2.2}$$

where $\alpha \in (0,1)$ is fractional order and $\varphi(t)$ is a real valued smooth function and a is real constant. According to the introduced modified Mittag-Leffler function the solution of equation (2.1) is looking for as following series:

$$u(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{mn}\frac{x^{m\alpha}}{(m\alpha)!}\frac{y^{n\alpha}}{(n\alpha)!}, \tag{2.3}$$

where u_{mn} are unknown coefficients. Let's suppose $(-k\alpha)! = \infty$, for $k \in \mathbb{N}$. According to the α-order derivative of Mittag-Leffler function, we have

$$D_x^\alpha u(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{mn}\frac{x^{(m-1)\alpha}}{[(m-1)\alpha]!}\frac{y^{n\alpha}}{(n\alpha)!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{(m+1)n}\frac{x^{m\alpha}y^{n\alpha}}{(m\alpha)!(n\alpha)!}, \tag{2.4}$$

and

$$D_y^\alpha u(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{mn}\frac{x^{(m\alpha)}}{(m\alpha)!}\frac{y^{(n-1)\alpha}}{[(n-1)\alpha]!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u_{m(n+1)}\frac{x^{m\alpha}y^{n\alpha}}{(m\alpha)!(n\alpha)!}. \tag{2.5}$$

By replacing these amounts in equation (2.1), we get the following recurrence (difference) equation:

$$u_{(m+1)n} = u_{m(n+1)}, \quad m,n \geqslant 0, \tag{2.6}$$

by imposing boundary conditions (2.2) to series solution (2.3) we obtain:

$$\sum_{m=0}^{\infty} u_{m0}\frac{t^{m\alpha}}{(m\alpha)!} = a\sum_{n=0}^{\infty} u_{0n}\frac{t^{n\alpha}}{(n\alpha)!} + \varphi(t), \quad t \geqslant 0, \tag{2.7}$$

or

$$\sum_{k=0}^{\infty}(u_{k0} - au_{0k})\frac{t^{k\alpha}}{(k\alpha)!} = \varphi(t), \quad t \geqslant 0, \tag{2.8}$$

from the difference equation (2.4) we see that

$$u_{mn} = u_{(m+n)0} = u_{0(m+n)}, \quad m,n \geqslant 0, \tag{2.9}$$

by using (2.6) and (2.5) we have:

$$\sum_{k=0}^{\infty} u_{k0}(1-a)\frac{t^{k\alpha}}{(k\alpha)!} = \varphi(t), \tag{2.10}$$

or

$$\sum_{k=0}^{\infty} u_{k0} \frac{t^{k\alpha}}{(k\alpha)!} = \frac{\varphi(t)}{1-\alpha}, \quad t \geqslant 0, \tag{2.11}$$

specially, for $t = 0, u_{00} = \frac{\varphi(0)}{1-\alpha}$. From (2.7) for some finite values of the fractional derivative of $k \in \mathbb{N}$, the $\alpha-$fractional derivative terms get

$$u_{00} \frac{t^{-\alpha}}{(-\alpha)!} + u_{10} \frac{1}{0!} + u_{20} \frac{t^{\alpha}}{\alpha!} + u_{30} \frac{t^{2\alpha}}{(2\alpha)!} + ... = \frac{1}{1-\alpha} D^{\alpha} \varphi(t), \tag{2.12}$$

specially for $t = 0$, we have:

$$u_{10} = \frac{1}{1-\alpha} D^{\alpha} \varphi(t)|_{t=0}, \tag{2.13}$$

by continuing this process, we get the following relation

$$u_{k0} = \frac{1}{1-\alpha} D^{k\alpha} \varphi(t)|_{t=0}, \quad k \geqslant 0. \tag{2.14}$$

To sum up, we conclude the following theorem:

Theorem 2.1. *Suppose a, α are real known constants and $\alpha \in (0,1)$. Also $\varphi(t)$ is a infinite differentiable function and satisfies in the following condition:*

$$\exists \quad 0 < M \in \mathbb{R}; \quad \left| D^{k\alpha} \varphi(t)|_{t=0} \right| \leqslant M, \quad k \geqslant 0. \tag{2.15}$$

Then the problem (2.1)-(2.2) has a unique solution in the form of (2.3).

Remark 2.2. According to M-test Weierstrass, convergence of series solution

$$u(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{1-\alpha} D^{(m+n)\alpha} \varphi(t)|_{t=0} \frac{x^{m\alpha}}{(m\alpha)!} \frac{y^{n\alpha}}{(n\alpha)!}, \tag{2.16}$$

is guaranteed, because its majorant series is converge for all fixed values of $x > 0$ and $y > 0$, that is

$$|u(x,y)| \leq \frac{M}{1-\alpha} \sum_{m=0}^{\infty} \frac{x^{m\alpha}}{(m\alpha)!} \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{(n\alpha)!}, \tag{2.17}$$

is converge. Because of factorial terms $(m\alpha)!$ and $(n\alpha)!$ in denominator.

3. BVP for FPDE in Bounded Domain

We study the following BVP for FPDE in a bounded domain as follows:

$$D_x^{\sqrt{3}} u(x,y) = D_y^{\sqrt{2}} u(x,y), \quad x \in (0,1), y \in (0,1), \tag{3.1}$$

with non-local boundary condition:

$$\alpha_0 u(0,t) + \alpha_1 u(1,t) + \beta_0(t,0) + \beta_1 u(t,1) = \varphi(t); \quad t \in [0,1], \tag{3.2}$$

α_i and $\beta_i, i = 0,1$ are known real constants and $\varphi(t)$ is known real valued continuous function. According to modified Mittag-Leffler function in section

1, we search the solution of the equation (3.1) in the form of the following series:

$$u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}\frac{x^{-1+(m-1)\sqrt{3}}}{[-1+(m-1)\sqrt{3}]!}\frac{y^{-1+n\sqrt{2}}}{(-1+n\sqrt{2})!}. \tag{3.3}$$

Note that the function

$$h_{\frac{1}{n}}(x)=\frac{x^{\frac{1-n}{n}}}{(\frac{1-n}{n})!}+\frac{x^{\frac{2-n}{n}}}{(\frac{2-n}{n})!}+...=\sum_{k=1}^{\infty}\frac{x^{\frac{k-n}{n}}}{(\frac{k-n}{n})!},\quad (-1)!=\infty, \tag{3.4}$$

is invariant to the fractional order $\alpha=\frac{1}{n}$ step derivative. Therefore we have

$$D_x^{\sqrt{3}}u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}\frac{x^{-1+(m-1)\sqrt{3}}}{[-1+(m-1)\sqrt{3}]!}\frac{y^{-1+n\sqrt{2}}}{(-1+n\sqrt{2})!} \tag{3.5}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{(m+1)n}\frac{x^{-1+m\sqrt{3}}}{[-1+m\sqrt{3}]!}\frac{y^{-1+n\sqrt{2}}}{(-1+n\sqrt{2})!}, \tag{3.6}$$

and

$$D_y^{\sqrt{2}}u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{m(n+1)}\frac{x^{-1+m\sqrt{3}}}{[-1+m\sqrt{3}]!}\frac{y^{-1+n\sqrt{2}}}{(-1+n\sqrt{2})!}. \tag{3.7}$$

By replacing these amounts in the equation(3.1), we get the following recurrence relation

$$u_{(m+1)n}=u_{m(n+1)}\quad ,m,n=1,2,... \tag{3.8}$$

or we have

$$u_{mn}=u_{m+n-1},\quad m,n=1,2,.... \tag{3.9}$$

By imposing the boundary condition (3.2) in a simple case that, $\alpha_0=\beta_0=0$.

$$\alpha_1u(1,t)+\beta_1u(t,1)=\varphi(t),\quad \alpha_1,\beta_1\in\mathbb{R}, \tag{3.10}$$

we obtain the following relation:

$$\alpha_1\sum_{k=1}^{\infty}u_{k1}\sum_{m=1}^{k}\frac{1}{(-1+m\sqrt{3})!}\frac{t^{-1+(k+1-m)\sqrt{2}}}{[-1+(k+1-m)\sqrt{2}]!}+ \tag{3.11}$$

$$\beta_1\sum_{k=1}^{\infty}u_{k1}\sum_{m=1}^{k}\frac{t^{-1+m\sqrt{3}}}{(-1+m\sqrt{3})!}\frac{1}{[-1+(k+1-m\sqrt{2}]!}=\varphi(t), \tag{3.12}$$

or

$$\sum_{k=1}^{\infty}u_{k1}v_k(t)=\varphi(t),\quad t\in[0,1], \tag{3.13}$$

where

$$v_k(t)=\sum_{m=1}^{\infty}\{\frac{\alpha_1}{-1+m\sqrt{3}}\frac{t^{-1+(k+1-m)\sqrt{2}}}{[-1+(k+1-m)\sqrt{2}]} \tag{3.14}$$

$$+\frac{\beta_1}{[-1+(k+1-m)\sqrt{2}]}\frac{t^{-1+m\sqrt{3}}}{-1+m\sqrt{3}}\},\quad k=1,2,.... \tag{3.15}$$

Suppose the function $w_k(t)$ is biorthogonal to $v_k(t)$, that is:

$$(v_k, w_s) = \delta_{ks} = \begin{cases} 1, & k = s \\ 0, & k \neq s \end{cases} \quad k, s = 1, 2, ..., \tag{3.16}$$

then from (3.5) and (3.6) we will have

$$\sum_{k=1}^{\infty} u_{k1}(v_k, w_s) = (\varphi, w_s), \quad s = 1, 2, ... \tag{3.17}$$

and also $u_{s1} = (\varphi, w_s), s = 1, 2, ...$. Finaly by using (3.7), we get the following series for the solution of problem (3.1)-(3.2):

$$u(x, y) = \sum_{k=1}^{\infty} (\varphi, w_s) \sum_{m=1}^{k} \frac{x^{-1+m\sqrt{3}}}{(-1 + m\sqrt{3})!} \frac{y^{-1+(k+1-m)\sqrt{2}}}{[-1 + (k + 1 - m)\sqrt{2}]!}. \tag{3.18}$$

To sum up we conclude the following theorem:

Theorem 3.1. *If α_1, β_1 are real constants, and $\varphi(t)$ is a continuous real valued function, and the functions $w_s(t)$ are biorthogonal to $v_k(t)$ in the form of (3.6), then the BVP (3.1)-(3.4) has a unique solution in form of (3.8).*

4. Analogy of Fractional Cauchy-Riemann Equation

In this section, we consider a FPDE which it is the fractional analogy of the Cauchy-Riemann equation. In fact, this equation is derived from decomposition of Laplace equation

$$(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_1^2})u(x) = (\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1})(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_1})u(x), \tag{4.1}$$

and

$$(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_1})u(x) = (D_2^{\frac{1}{2}} + \frac{1-i}{\sqrt{2}} D_1^{\frac{1}{2}})(D_2^{\frac{1}{2}} - \frac{1-i}{\sqrt{2}} D_1^{\frac{1}{2}})u(x). \tag{4.2}$$

Therefore we consider a BVP consists of the analogy of fractional Cauchy-Riemann equation

$$D_2^{\frac{1}{2}} u(x) + aD_1^{\frac{1}{2}} u(x) = 0, \quad 0 < x_j < 1, j = 1, 2, \tag{4.3}$$

with boundary condition

$$u(t, 0) + \alpha u(0, t) + \beta u(t, 1) + \gamma u(1, t) = \varphi(t), \quad 0 \leq t \leq 1, \tag{4.4}$$

where α, β, γ, are complex constants and $\varphi(t) \in \mathbb{C}[0, 1]$. We search the solution of equation (4.1) in the following form

$$u(x) = u(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn} \frac{x_1^{\frac{m}{2}}}{(\frac{m}{2})!} \frac{x_2^{\frac{n}{2}}}{(\frac{n}{2})!}. \tag{4.5}$$

We suppose $(\frac{-1}{2})! = \infty$. Then by derivating of proposed solution (4.3) and considering some relations in pervious sections we will have the recurrence relation

$$u_{m(n+1)} + au_{(m+1)n} = 0, \quad m \geqslant 0, n \geqslant 0, \tag{4.6}$$

or

$$u_{m(n+1)} = (-a)^n u_{(m+1)n}, \quad m \geqslant 0, n \geqslant 0. \tag{4.7}$$

By substituting in boundary condition (4.2) we get:

$$\sum_{m=0}^{\infty} u_{m0} \frac{t^{\frac{m}{2}}}{(\frac{m}{2})!} + \alpha \sum_{n=0}^{\infty} u_{0n} \frac{t^{\frac{n}{2}}}{(\frac{n}{2})!} + \sum_{m=0}^{\infty} \frac{t^{\frac{m}{2}}}{(\frac{m}{2})!} \sum_{n=0}^{\infty} u_{mn} \frac{1}{(\frac{n}{2})!} \tag{4.8}$$

$$+ \sum_{n=0}^{\infty} \frac{t^{\frac{n}{2}}}{(\frac{n}{2})!} \sum_{m=0}^{\infty} u_{mn} \frac{1}{(\frac{m}{2})!} = \varphi(t). \tag{4.9}$$

Suppose the function $\varphi(t)$, has a series expansion as follows:

$$\varphi(t) = \sum_{k=0}^{\infty} \varphi_k \frac{t^{\frac{k}{2}},}{(\frac{k}{2})!} \tag{4.10}$$

then we have:

$$[1 + \alpha(-a)^m] u_{m0} \tag{4.11}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{(\frac{n}{2})!} [\beta(-a)^n + \gamma(-a)^m] u_{(m+n)0} = \varphi_m, \quad m \geqslant 0, \tag{4.12}$$

If the coefficient of u_{m0} is unvanish, then we can determine the arbitrary coefficients of u_{k0} of series solution (4.3). Finally we will have the following final theorem.

Theorem 4.1. *If α, β, γ are complex constsnts and $\varphi(t)$ is complex valued function, and the expansion (4.5) and the condition*

$$1 + \alpha(-a)^m + \beta + \gamma(-a)^m \neq 0, \quad m \geqslant 0, \tag{4.13}$$

are valid, then the BVP (4.1)-(4.2) has a unique solution in form of (4.3)

Remark 4.2. In the special case of boundary condition (4.2), $\beta = \gamma = 0$ then under condition

$$1 + \alpha(-a)^m \neq 0, \tag{4.14}$$

we get the following results:

$$u_{k0} = \frac{\varphi_k}{1 + \alpha(-a)^k}; \quad k \geqslant 0, \tag{4.15}$$

and the normal solution for the problem (4.1)-(4.2) is

$$u(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x_1^{\frac{m}{2}}}{(\frac{m}{2})!} \frac{x_2^{\frac{n}{2}}}{(\frac{n}{2})!} (-a)^n \frac{\varphi_{m+n}}{1 + \alpha(-a)^{m+n}}. \tag{4.16}$$

Therefore this formal solution is converge when the following conditions are satisfied:

$$|a| \leq 1, |\varphi_k| < M, \quad k \geqslant 0. \tag{4.17}$$

5. Conclusion

In this paper, some BVP involving FPDE were presented. By means of modified Mittag-Lefller functions we presented their solutions in the form of series. Then by using the theory of ordinary differential equations and the concept of fractional step, we extended the solving process to the general and advanced cases. Finally, regarding the general terms of series solutions (1.4), (2.3), (3.8) and (4.3) which include factorial terms, the convergency of sereis solution is guaranteed. Conditions of existence and uniquencees of solution for problem (2.1)-(2.2) and problem (3.1)-(3.2) the as three theorem were expressed.

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