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A Bound for the Nilpotency Class of a Lie Algebra

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Abstract. In the present paper, we compare the nilpotency class $N(L)$ of a nilpotent Lie algebra L with that of its proper subalgebras. As the main result, we prove that $N(L) \leq \lfloor nd/(d-1) \rfloor$ where $n = \max\{N(S) : S$ is a proper subalgebra of $L\}$, d is the minimal number of generators of L and $\lfloor\ \rfloor$ denotes the integral part.

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1. Introduction

Nilpotent Lie algebras are very important in the classification theory of Lie algebras, where they play a central role as a consequence of the Levi theorem and the works of Malcev. The first significant research about nilpotent Lie algebras is due to Umlauf in the 19th century. In the 40's and 50's, Morozov and Dixmier began with the systematical study of this class of algebras (see [1, 6] for more details). The structure of a nilpotent Lie algebra (group) with regards to its subalgebras (subgroups) has been studied by some authors (see [2, 4] for instance).

As the main theorem, we discuss a bound for the nilpotency class of a Lie algebra with respect to the nilpotency class of its subalgebras. We actually prove that if L is a nilpotent Lie algebra whose proper subalgebras are all nilpotent of class at most n, then the class of L is at most $\lfloor nd/(d-1) \rfloor$, where

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d is the minimal number of generators of L and $\lfloor \ \rfloor$ denotes the integral part. As a consequence, we show that the *Heisenberg Lie algebra* of dimension three is the only finite dimensional non-abelian nilpotent Lie algebra whose proper subalgebras are all abelian.

Throughout of this paper, all Lie algebras are over a fixed field Λ and L^n denotes the nth term of the lower central series of a Lie algebra L, defined inductively by $L^1 = L$ and $L^n = [L^{n-1}, L]$, for $n \geq 2$, where $[,]$ denotes the Lie bracket.

2. Main Results

In this section, we discuss some preliminary known results and then prove the main theorem.

Let L be a Lie algebra and H be a subalgebra of L. Then the idealizer of H in L is defined to be

$$I_L(H) = \{x \in L | [x, y] \in H, \forall y \in H\}.$$

One can easily see that $I_L(H) = L$ if and only if H is an ideal of L. The following is an immediate result of the above definition.

Lemma 2.1. *If L is a Lie algebra and H, K are two subalgebras of L, then $[H, K] \leq K$ if and only if $H \leq I_L(K)$.*

Lemma 2.2. *Let L be a nilpotent Lie algebra and H be a proper subalgebra of L. Then $H \neq I_L(H)$.*

Proof. See [5], p. 14. □

The above results can easily imply the following proposition.

Proposition 2.3. *Let L be a nilpotent Lie algebra and H be a maximal subalgebra of L. Then H is an ideal of L.*

Now, we are ready to prove our main results.

Theorem 2.4. *Let L be a nilpotent Lie algebra of class $N(L)$. Then $N(L) \leq \lfloor nd/(d-1) \rfloor$ where $n = \max\{N(S) : S$ is a proper subalgebra of $L\}$, $d > 1$ is the minimal number of generators of L and $\lfloor \ \rfloor$ denotes the integral part.*

Proof. Let $X = \{x_1, \ldots, x_d\}$ be a minimal set of generators of L and $m = \lfloor nd/(d-1) \rfloor$. It is sufficient to show that $[y_1, y_2, \ldots, y_{m+1}] = 0$, where every $y_i \in X$. Let $\ell = \lfloor n/(d-1) \rfloor$. Then $m = n + \ell$ and so $m+1 < (\ell+1)d$, which implies that not all elements of X can occur more than ℓ times in $[y_1, y_2, \ldots, y_{m+1}]$. Therefore, there exists an element x_1 say, which appears at most ℓ times in this Lie bracket. Since $x_2, x_3, \ldots, x_d$ do not generate L, there is a maximal subalgebra H of L which contains $x_2, x_3 \ldots, x_d$. By Proposition 2.3, H is an ideal of L. Now $[y_1, y_2, \ldots, y_{m+1}]$ contains at least $m + 1 - \ell = n + 1$ elements of H. Now, it is sufficient to show that $[y_1, y_2, \dots, y_{m+1}] \in H^{n+1}$, since H is nilpotent of class at most n. If $[y_1, y_2, \dots, y_{m+1}]$ contains at least one element of H, then clearly $[y_1, y_2, \dots, y_{m+1}] \in H$. Assume that the result is true for every positive integer less than $n+1$. If $y_{m+1} \in H$, then $[y_1, y_2, \dots, y_m]$ contains at least n elements of H and hence it belongs to H^n and so $[y_1, y_2, \dots, y_{m+1}] \in H^{n+1}$. If $y_{m+1} \notin H$, then $[y_1, y_2, \dots, y_m]$ has already at least $n+1$ elements of H. Now, if $y_m \in H$, then $[y_1, y_2, \dots, y_{m-1}]$ contains at least n elements of H and by a similar manner $[y_1, y_2, \dots, y_{m-1}] \in H^n$ and so $[y_1, y_2, \dots, y_{m+1}] = 0$. If $y_m \notin H$, then continuing this proceeding completes the proof. □

The following are immediate corollaries of Theorem 2.4.

Corollary 2.5. *If L is a nilpotent Lie algebra whose proper subalgebras are all nilpotent of class at most n and $d(L) > n+1$, then L has also nilpotency class at most n.*

Corollary 2.6. *If L is a nilpotent Lie algebra of class $2n$, whose proper subalgebras are all nilpotent of class at most n, then $d(L) = 2$.*

It is easy to see that the Heisenberg Lie algebra and n-dimensional abelian Lie algebra ($n \geq 3$), are Lie algebras satisfying Theorem 2.4. Recall that the Heisenberg Lie algebra $H(m)$ is the $2m+1$ dimensional real Lie algebra with the basis $\{a_1, \dots, a_m, b_1, \dots, b_m, c\}$ and the Lie brackets defined by

$$[a_i, a_j] = [b_i, b_j] = [a_i, c] = [b_i, c] = [c, c] = 0 \; and \; [a_i, b_j] = c\delta_{ij},$$

where δ_{ij} is the Kronecker delta (see [3] for more details).

In the following result, we show that there is only one finite dimensional non-abelian nilpotent Lie algebra whose proper subalgebras are all abelian.

Corollary 2.7. *If L is an r-dimensional non-abelian nilpotent Lie algebra whose proper subalgebras are all abelian, then $r = 3$ and $L \cong H(1)$.*

Proof. By using Theorem 2.4, we have $\lfloor \frac{d}{d-1} \rfloor = m$. If $d \geq 3$, then L is abelian which is a contradiction. Therefore, $d = m = 2$ and hence $L^2 \leq Z(L)$. Also if $1 \leq r \leq 2$, then either L is abelian or L has a basis $\{x, y\}$ say, in which $[x, y] = x$. In the latter situation, the center of L is trivial (see [3], Theorem 3.1) and hence L is not nilpotent. Therefore, $r \geq 3$. Since $d = 2$, one can easily check that $dimL^2 = 1$ and also $r = 3$. Hence $L \cong H(1)$ (see [3], p. 21). □

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