Copresented Dimension of Modules

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Abstract. In this paper, a new homological dimension of modules, copresented dimension, is defined. We study some basic properties of this homological dimension. Some ring extensions are considered, too. For instance, we prove that if \( S \geq R \) is a finite normalizing extension and \( S_R \) is a projective module, then for each right \( S \)-module \( M_S \), the copresented dimension of \( M_S \) does not exceed the copresented dimension of \( \text{Hom}_R(S,M) \).

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1. Introduction

Throughout this paper, \( R \) is an associative ring with identity and all modules are unitary. First we recall some known notions and facts needed in the sequel. Let \( R \) be a ring, \( n \) a non-negative integer and \( M \) an \( R \)-module. Then

1. \( M \) is said to be \textit{finitely cogenerated} [1] if for every family \( \{V_k\}_J \) of submodules of \( M \) with \( \bigcap_j V_k = 0 \), there is a finite subset \( I \subset J \) with \( \bigcap_I V_K = 0 \).

2. \( M \) is said to be \textit{n-copresented} [14] if there is an exact sequence of \( R \)-modules \( 0 \to M \to E^0 \to E^1 \to \cdots \to E^n \), where each \( E^i \) is a finitely cogenerated injective module.

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(3) $R$ is called right co-coherent [17] if every finitely cogenerated factor module of a finitely cogenerated injective $R$-module is finitely copresented.

(4) $R$ is called $n$-cocoherent [14] in case every $n$-copresented $R$-module is $(n + 1)$-copresented. It is easy to see that $R$ is cocoherent if and only if it is 1-cocoherent. Recall that a ring $R$ is called right co-noethrian [4] if every factor module of a finitely cogenerated $R$-module is finitely cogenerated. By [4, Proposition 17], a ring $R$ is co-noethrian if and only if it is 0-cocoherent.

(5) $M$ is said to be $n$-presented [5] if there is an exact sequence of $R$-modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$, where each $F_i$ is a finitely generated free module.

(6) $R$ is called coherent [18] in case every 0-presented $R$-module is 1-presented.

(7) A ring extension $R \subseteq R'$ with characteristic $p > 0$ is called a purely inseparable extension [10] if for every element $r' \in R'$, there exists a non-negative integer $n$ such that $r'^p \in R$.

(8) For any commutative ring $R$ of prime characteristic $p > 0$, assume that $F_R : R \rightarrow R^{(e)}$ is the $e$-th iterated Frobenius map in which $R^{(e)} \cong R$. Then, the perfect closure [9] of $R$, denoted by $R^\infty$, is defined as the limit of the following direct system:

$$
\begin{array}{cccccccccccc}
R & \xrightarrow{F_R} & R & \xrightarrow{F_R} & R & \xrightarrow{F_R} & \cdots \\
\end{array}
$$

(9) $M$ is called $(n, d)$-injective [18] if $\text{Ext}^d_{R^n}(N, M) = 0$ for any $n$-presented right $R$-module $N$. It is clear that $M$ is $(0, 0)$-injective if and only if $M$ is injective.

(10) Assume that $S \geq R$ is a unitary ring extension. Then, the ring $S$ is called right $R$-projective [6] in case, for any right $S$-module $M_S$ with an $S$-module $N_S$, $N_R | M_R$ implies $N_S | M_S$, where $N | M$ means that $N$ is a direct summand of $M$.

(11) The ring extension $S \geq R$ is called a finite normalizing extension [8] in case there is a finite subset $\{s_1, \cdots, s_n\} \subseteq S$ such that $S = \sum_{i=1}^{n} s_i R$ and $s_i R = Rs_i$ for $i = 1, \cdots, n$.

(12) A finite normalizing extension $S \geq R$ is called an almost excellent extension [12] in case $\mathcal{R}S$ is flat, $\mathcal{S}_R$ is projective, and the ring $S$ is right $R$-projective.

In this paper, we introduce the dual concepts of presented dimensions of $R$-modules. We also, introduce the copresented dimension of any $R$-module $M$:

$$
\text{Fed}(M) = \inf\{m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^m \rightarrow \cdots \rightarrow E^{m+i} \rightarrow \cdots, \text{ such that } E^{m+i} \text{ are finitely cogenerated for }
$$
If $K = \ker(E^m \to E^{m+1})$, then $K$ has an infinite finite copresentation. It is clear that any copresented dimension is finitely copresented dimension (see [16]). Also, the copresented dimension of ring $R$ is defined to be:

$$\text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}.$$ 

Then, some basic properties of the copresented dimensions of modules are studied. For example, it is shown that if $\text{FEd}(M) < \infty$, then $\text{id}(M) \leq n$ if and only if $\text{Ext}^n_R(N, M) = 0$ for every strongly copresented $R$-module $N$. Also, it is proved that $\text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}$, for any two rings $R$ and $S$. Also, some characterizations of the copresented dimensions of modules on Ring Extensions are determined. For instance, let $S \geq R$ be a finite normalizing extension with $S_R$ projective as an $R$-module, then for any right $R$-module $M_R$, we have $\text{FEd}(\text{Hom}_R(S, M)) \leq \text{FEd}(M_R)$. Finally, we give a sufficient condition under which $\text{FED}(S) \leq \text{FED}(R)$ and $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$, where $k = \text{id}(S_R)$ and $d = \sup\{\text{FEd}(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FEd}(M_S) = 0\}$.

2. **Main Results**

We start this section with the following definition which is the dual of the presented dimension of a module.

**Definition 2.1.** For any $R$-module $M$, we define the copresented dimension of $M$ to be $\text{FEd}(M) = \inf\{m \mid$ there exists an injective resolution $0 \to M \to E^0 \to \cdots \to E^m \to \cdots \to E^{m+i} \to \cdots$, so that $E^{m+i}$ are finitely cogenerated for $i = 0, 1, 2, \cdots \}$. In particular, a module $M$ is called strongly copresented module if $\text{FEd}(M) = 0$.

**Proposition 2.2.** For any $R$-module $M$, $\text{FEd}(M) \leq \text{id}(M) + 1$.

**Proof.** It is a direct consequence of Definition 2.1. □

**Example 2.3.** Let $R = \mathbb{Z}$. Since $\text{id}(\mathbb{Z}_{p^\infty}) = 0$, we have $\text{FEd}(\mathbb{Z}_{p^\infty}) \leq 1$. On the other hand, $\mathbb{Z}_{p^\infty}$ is finitely cogenerated by [1, p.124]. So by Definition 2.1, $\text{FEd}(\mathbb{Z}_{p^\infty}) = 0$.

Now, we study the behavior of the copresented dimension on the exact sequences. Before this we need the following lemma.

**Lemma 2.4.** Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of $R$-modules. Then:

1. If $0 \to A \to A^0 \to A^1 \to \cdots$ and $0 \to C \to C^0 \to C^1 \to \cdots$ are injective resolutions of $A$ and $C$, respectively. Then the exact sequence

$$0 \to B \to A^0 \oplus C^0 \to A^1 \oplus C^1 \to \cdots$$

is an injective resolution of $B$. 


(2) If $0 \to B \to B^0 \to B^1 \to \cdots$ and $0 \to C \to C^0 \to C^1 \to \cdots$ are injective resolutions of $B$ and $C$, respectively. Then the exact sequence

$$0 \to A \to B^0 \to D^0 \to D^1 \to \cdots$$

is an injective resolution of $A$, where $D^i = C^i \oplus B^{i+1}$ for any $i \geq 0$.

(3) If $0 \to B \to B^0 \to B^1 \to \cdots$ and $0 \to A \to A^0 \to A^1 \to \cdots$ are injective resolutions of $B$ and $A$, respectively. Then the exact sequence

$$0 \to C \to F^0 \to E^0 \to E^1 \to \cdots$$

is an injective resolution of $C$, where $F^0 = B^0 \oplus A^1$ and $E^i = A^0 \oplus B^{i+1} \oplus A^{i+2}$ for any $i \geq 0$.

Proof. (1) The proof is similar to that of [3, Theorem 2.4].

(2) Let $0 \to B \to B^0 \to B^1 \to \cdots$ be an injective resolution of $B$. Then, the exact sequences

$$0 \to K \to B^1 \to B^2 \to \cdots$$

and

$$0 \to B \to B^0 \to K \to 0$$

exist, where $K = \frac{B^0}{\mathfrak{I}}$. Now, we consider the following commutative diagram:

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & A & B & C & 0 \\
\| & \downarrow & \downarrow & \downarrow \\
0 & A & B^0 & D & 0 \\
\downarrow & \downarrow & \downarrow \\
K & K \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

By (1), there is an exact sequence

$$0 \to D \to D^0 \to D^1 \to D^2 \to \cdots$$

of injective $R$-modules $D^i$ such that $D^i = C^i \oplus B^{i+1}$ for any $i \geq 0$.

Combining this sequence with the exact sequence $0 \to A \to B^0 \to D \to 0$, we get the exact sequence

$$0 \to A \to B^0 \to D^0 \to D^1 \to \cdots$$

where $B^0$ and $D^i$ are injective for any $i \geq 0$.

(3) Let $0 \to A \to A^0 \to A^1 \to \cdots$ be an injective resolution of $A$. Then, the exact sequences

$$0 \to K \to A^1 \to A^2 \to \cdots$$

and

$$0 \to A \to A^0 \to K \to 0$$

exist, where $K = \frac{A^0}{A}$. Now, we consider the following commutative diagram:
By (1), there is an exact sequence
\[ 0 \rightarrow F \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots \]
of injective \( R \)-modules \( F^i \) such that \( F^i = B^i \oplus A^{i+1} \) for any \( i \geq 0 \).

It is clear that \( F = A^0 \oplus C \). So, the exact sequence \( 0 \rightarrow C \rightarrow F \rightarrow A^0 \rightarrow 0 \) exists. Let \( K = F^0 / F \), then we obtain the following commutative diagram:

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & A & B & C & 0 \\
\downarrow & \downarrow & \parallel \\
0 & A^0 & F & C & 0 \\
\downarrow & \downarrow \\
K &=& K \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

Therefore by (1), the sequence
\[ 0 \rightarrow E \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \]
is an injective resolution of \( E \), where \( E^i = A^0 \oplus F^i+1 = A^0 \oplus B^{i+1} \oplus A^{i+2} \) for any \( i \geq 0 \). Combining this sequence with the exact sequence \( 0 \rightarrow C \rightarrow F^0 \rightarrow E \rightarrow 0 \), we get the exact sequence
\[ 0 \rightarrow C \rightarrow F^0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots , \]
where \( F^0 \) and \( E^i \) are injective for any \( i \geq 0 \). \( \Box \)

**Theorem 2.5.** Let \( 0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0 \) be an exact sequence of \( R \)-modules. Then \( \text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\} \), \( \text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\} \), \( \text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\} \).

**Proof.** Assume that \( E^* \) is an injective resolution of \( A \) and \( E'^* \) is an injective resolution of \( C \). Thus by Lemma 2.5(1), there exists an injective resolution \( E \) of \( B \) such that
\[ 0 \rightarrow E^A \rightarrow E^B = E^A \oplus E'^C \rightarrow E'^C \rightarrow 0 \]
is an exact sequence of complexes. Hence for every $m \geq \max\{\text{FEd}(A), \text{FEd}(C)\}$, $E^m$ is finitely cogenerated. So, we deduce that $\text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\}$.

Assume that $E^-$ is an injective resolution of $C$ and $E$ is an injective resolution of $B$. Thus by Lemma 2.5(2), the exact sequence

$$0 \to A \to E^0 \to D^0 \to D^1 \to \cdots \to D^d \to \cdots$$

is an injective resolution of $A$. So for every $d \geq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$, $D^d$ is finitely cogenerated. Thus, we have that $\text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$. Also, it is prove that $\text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\}$. \hfill \qed

The proof of the following Corollary is similar to the proof of [19, Corollary 2.7].

**Corollary 2.6.** If $\text{FEd}(M_1), \text{FEd}(M_2), \cdots \text{FEd}(M_d)$ are finite, then:

$$\text{FEd}(\oplus M_i) = \max\{\text{FEd}(M_i) \mid i = 1, \cdots, d\}.$$  

*Proof.* For the case $m = 2$, the exact sequences

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$$

and

$$0 \to M_2 \to M_2 \oplus M_1 \to M_1 \to 0$$

exist. Thus by Theorem 2.5, we deduce that

$$\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_1) - 1\},$$

$$\text{FEd}(M_1) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 1\}$$

and

$$\text{FEd}(M_1 \oplus M_2) \leq \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}.$$  

Assume that $\text{FEd}(M_1) < \text{FEd}(M_2)$. Then $\text{FEd}(M_1) \leq \text{FEd}(M_2) - 1$, and we have:

$$\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 2\} = \text{FEd}(M_1 \oplus M_2).$$

Also, similarly $\text{FEd}(M_1) \leq \text{FEd}(M_1 \oplus M_2)$. So, we conclude that $\text{FEd}(M_1 \oplus M_2) = \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}$. \hfill \qed

**Proposition 2.7.** Let $n$ be a non-negative integer. Then the following statements are equivalent:

1. $\text{id}(M) \leq n$ for every strongly copresented $R$-module $M$;
2. $\text{Ext}^{n+1}_R(N, M) = 0$ for every strongly copresented $R$-module $N$.

*Proof.* (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) We use the induction on $n$. Let $n = 0$. Since $\text{Ext}^1_R(N, M) = 0$ for any strongly copresented $R$-module $N$, by using the exact sequence $0 \to M \to E^0 \to L^0 \to 0$ where $E^0$ is finitely cogenerated and $L^0$ is strongly copresented, we deduce that $\text{Ext}^1_R(L^0, M) = 0$. Therefore by [7, Theorem 7.31], the exact sequence above is split. So, $M$ is injective and hence $\text{id}(M) \leq 0$. Assume that
\( n > 0. \) By [7, Corollary 6.42], we have that \( \text{Ext}_R^m(N, M) \cong \text{Ext}_R^n(N, L^0) = 0. \) Thus by induction hypothesis, \( \text{id}(L^0) \leq n - 1. \) Therefore from the exact sequence above, we deduce that \( \text{id}(M) \leq n. \)

**Proposition 2.8.** Let \( \text{FEd}(M) \leq 1. \) Then the following statements are equivalent:

1. \( \text{id}(M) \leq n; \)
2. \( \text{Ext}_R^{n+1}(N, M) = 0 \) for every strongly copresented \( R \)-module \( N. \)

**Proof.** Since \( \text{FEd}(M) \leq 1, \) the exact sequence \( 0 \rightarrow M \rightarrow E^0 \rightarrow L^0 \rightarrow 0 \) exists, where \( E^0 \) is injective and \( L^0 \) is strongly copresented. Thus, \( \text{Ext}_R^{n+1}(N, M) = 0 \) for any strongly copresented \( R \)-module \( N \) if and only if \( \text{Ext}_R^n(N, L^0) = 0 \) if and only if \( \text{id}(L^0) \leq n - 1 \) (by Proposition 2.7) if and only if \( \text{id}(M) \leq n. \)

**Theorem 2.9.** Let \( \text{FEd}(M) < \infty. \) Then the following statements are equivalent:

1. \( \text{id}(M) \leq n; \)
2. \( \text{Ext}_R^{n+1}(N, M) = 0 \) for every strongly copresented \( R \)-module \( N. \)

**Proof.** (1) \( \Rightarrow \) (2) It is clear.

(2) \( \Rightarrow \) (1) If \( \text{FEd}(M) = m, \) then the exact sequence
\[
0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{m-1} \rightarrow E^m \rightarrow \cdots \rightarrow E^{m+i} \rightarrow \cdots
\]
eexists, where \( E^i \) is finitely cogenerated for any \( i \geq m. \) By Proposition 2.2, \( n + 1 \geq m. \) Let \( \text{Ext}_R^{n+1}(N, M) = 0 \) for every strongly copresented \( R \)-module \( N. \) Thus by [7, Corollary 6.42], we have
\[
\text{Ext}_R^{n+1}(N, M) \cong \text{Ext}_R^{n-m+1}(N, \text{coker}d^{m-1}) = 0.
\]
Since \( \text{coker}d^{m-1} \) is strongly copresented, Proposition 2.8 implies that
\[
\text{id}(\text{coker}d^{m-1}) \leq n - m
\]
and so, we deduce that \( \text{id}(M) \leq n. \)

**Corollary 2.10.** Let \( D(R) < \infty. \) Then:
\[
D(R) = \sup\{\text{pd}(N) \mid N \text{ is strongly copresented}\}.
\]

**Proof.** Assume that \( D(R) \leq m. \) Thus, \( \text{pd}(N') \leq m \) for any \( R \)-module \( N'. \) So, for any strongly copresented \( R \)-module \( N, \) \( \text{pd}(N) \leq m. \) Conversely, let \( \text{pd}(N) \leq m \) for every strongly copresented \( R \)-module \( N. \) Thus \( \text{Ext}_R^{m+1}(N, M) = 0 \) for every strongly presented \( R \)-module \( M. \) Since \( D(R) < \infty, \) \( \text{FEd}(M) < \infty \) by Proposition 2.2. Therefore by Theorem 2.9, \( \text{id}(M) \leq m \) and hence by [19, corollary 3.7], \( D(R) \leq m. \)

**Definition 2.11.** For any ring \( R, \) we define the copresented dimension of \( R \) to be \( \text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}. \)
Example 2.12. Let \( R = k[x^3, x^3 y, xy^3, y^3] \), where \( k \) is a field with characteristic \( p = 3 \). By Definition 2.11 and Proposition 2.2, \( \text{FED}(R^\infty) \leq D(R^\infty) + 1 \), where \( R^\infty \) is perfect closure of \( R \). On the other hand, \( k[x, y] \) is purely inseparable over \( R \). Also, by [9, Proposition 3.3], \((k[x, y])^\infty \) is coherent. Therefore by [10, Remark 1.4], \( R^\infty \) is coherent. Since \( R \) is reduced, [2, Proposition 5.5] implies that \( \text{FED}(R^\infty) \leq \dim(R) + 1 \) and so, \( \text{FED}(R^\infty) \leq 3 \).

Proposition 2.13. The following statements are equivalent:

1. \( \text{FED}(R) = 0 \);
2. Every finitely cogenerated module has an infinite finite copresented;
3. Every finitely cogenerated module is finitely copresented;
4. \( R \) is co-noetherian.

Proof. The implication (1) \( \implies \) (2) \( \implies \) (3) follow immediately from Definition 2.11.

(3) \( \implies \) (4) \( \implies \) (1) are trivial. \( \square \)

Corollary 2.14. If \( \text{FED}(R) \leq 0 \), then \( R \) is \( n \)-coherent.

Proof. Since every \( n \)-copresented module \( M \) is finitely cogenerated, Proposition 2.13 implies that \( M \) is \((n + 1)\)-copresented. \( \square \)

Next, we study the copresented dimension of the direct sum of rings. But before this we need the following lemma.

Lemma 2.15. Let \( f : R \to S \) be a ring epimorphism. If \( M_S \) is a right \( S \)-module (hence a right \( R \)-module) and \( N_R \) is a right \( R \)-module, then the following statements hold:

1. \( M \otimes_R S \cong M_S \).
2. If \( f \) is flat and \( N_R \) is a finitely cogenerated right \( R \)-module, then \( N \otimes_R S \) is a finitely cogenerated right \( S \)-module.
3. If \( f \) is flat, then \( M_S \) is a finitely cogenerated right \( S \)-module if and only if \( M_R \) is a finitely cogenerated right \( R \)-module.
4. If \( f \) is projective, then \( M_S \) is an injective right \( S \)-module if and only if \( M_R \) is an injective right \( R \)-module.

Proof. (1) This is clear.

(2) For any family of submodules \( \{ N_i \otimes_R 1_S \mid i \in I \} \) in \( N \otimes_R S \), if \( \bigcap_{i \in F} (N_i \otimes_R 1_S) = 0 \) for some finite subset \( F \) of \( I \). Since \( f \) is flat, we have that \( \bigcap_{i \in I} N_i \otimes_R 1_S = 0 \). So, \( \bigcap_{i \in F} N_i = 0 \) and hence by hypothesis \( \bigcap_{i \in F} N_i = 0 \) for some finite subset \( F \) of \( I \). Therefore, \( \bigcap_{i \in F} (N_i \otimes_R 1_S) = \bigcap_{i \in F} N_i \otimes_R 1_S = 0 \).

(3) \((\Rightarrow)\): Let \( \psi : M \to \prod_{i \in I} R \) is a monomorphism, then we claim that \( \pi : M \to \prod_{i \in F} R \) is a monomorphism for some finite subset \( F \) of \( I \). We have the following commutative diagram:
\[
\begin{align*}
M & \xrightarrow{\psi} \prod_{i \in I} R \\
\downarrow \cong & \quad \downarrow g \\
M & \xrightarrow{h} \prod_{i \in I} S,
\end{align*}
\]
where since \(g\) is epimorphism and \(\psi\) is monomorphism, \(h\) is monomorphism.
So by hypothesis, \(\alpha : M \to \prod_{i \in F} S\) is a monomorphism for some finite subset \(F\) of \(I\). Therefore the following commutative diagram:
\[
\begin{align*}
M & \xrightarrow{\gamma} \prod_{i \in F} R \\
\downarrow \cong & \quad \downarrow \beta \\
M & \xrightarrow{\alpha} \prod_{i \in F} S,
\end{align*}
\]
where \(\beta\) is epimorphism and \(\alpha\) is monomorphism, implies that \(\gamma\) is monomorphism.

\((\Leftarrow)\) : This follows from (1) and (2)

(4) By [5, Lemma 3.3], \(M_S\) is an \((n, d)\)-injective right \(S\)-module if and only if \(M_R\) is an \((n, d)\)-injective right \(R\)-module. If \(n = 0, d = 0\), Then (4) is hold. \(\square\)

**Theorem 2.16.** Assume that \(R\) and \(S\) are two rings. Then:

\[\text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}.\]

**Proof.** We first show that \(\text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\}\). Consider \(\text{FED}(R) = n, \text{FED}(S) = m\) and \(n \geq m\). Also, let \(M\) be a finitely cogenerated right \((R \oplus S)\)-module. Then \(M\) has a unique decomposition \(M = A \oplus B\), where \(A, B\) are right modules of rings \(R\) and \(S\), respectively. By [15, Lemma 1.1], \(A\) and \(B\) are finitely cogenerated right \((R \oplus S)\)-module. So by Lemma 2.15, \(A\) is finitely cogenerated right \(R\)-module and \(B\) is finitely cogenerated right \(S\)-module. Therefore \(\text{Fed}(A) \leq n\) and \(\text{Fed}(B) \leq m\), and hence there is an exact sequences

\[0 \to A \to E_a^0 \to E_a^1 \to \cdots \to E_a^{n-1} \to E_a^n \to \cdots,\]
\[0 \to B \to E_b^0 \to E_b^1 \to \cdots \to E_b^{m-1} \to E_b^m \to \cdots\]
of injective right \(R\)-modules \(E^i_a\) and injective right \(S\)-modules \(E^i_b\) such that \(E^i_a, E^i_b\) are finitely cogenerated for any \(i \geq n\) and \(i \geq m\), respectively. So, we deduce that the exact sequence

\[0 \to A \oplus B \to E_a^0 \oplus E_b^0 \to E_a^1 \oplus E_b^1 \to \cdots \to E_a^{n-1} \oplus E_b^{m-1} \to E_a^n \oplus E_b^m \to \cdots\]
even exists, where by Lemma 2.15, every \(E^i_a \oplus E^i_b\) is injective right \((R \oplus S)\)-module, and also, every \(E^i_a \oplus E^i_b\) is finitely cogenerated for any \(i \geq n\). Therefore, we have \(\text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\}\).

Conversely, Assume that \(\text{FED}(R \oplus S) = d\). If \(M\) is a finitely cogenerated right \(R\)-module. Then by Lemma 2.15, \(M\) is a finitely cogenerated right \((R \oplus S)\)-module and hence \(\text{FED}(M_{R \oplus S}) \leq d\). Thus, the exact sequence \(0 \to M \to E^0 \to E^1 \to \cdots \to E^{d-1} \to E^d \to \cdots\) of injective right \((R \oplus S)\)-modules
$E^i$ exists, where every $E^i$ is finitely cogenerated for any $i \geq d$. Let $E^i = C^i \oplus D^i$, where $C^i$ is a $R$-module and $D^i$ is a $S$-module. On the other hand, $M$ is a right $R$-module, so we have the exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^{d-1} \to C^d \to \cdots$ of $R$-modules. But, every $C^i$ is injective right $(R \oplus S)$-module and also every $C^i$ is finitely cogenerated right $(R \oplus S)$-module for $i \geq d$. So by [15, Lemma 1.1] and Lemma 2.15, $C^i$ is an injective right $R$-module and it is finitely cogenerated $R$-module for $i \geq d$. Therefore $\text{FEd}(M) \leq d$ and hence $\text{FED}(R) \leq d$. Similarly, $\text{FED}(S) \leq d$ and implies that $\text{sup}\{\text{FED}(R), \text{FED}(S)\} \leq \text{FED}(R \oplus S)$.

**Proposition 2.17.** Let $S \geq R$ be a finite normalizing extension with $S_R$ projective as an $R$-module. Then for any right $R$-module $M_R$, $\text{FEd}(\text{Hom}_R(S,M))_s \leq \text{FEd}(M_R)$.

**Proof.** Assume that $\text{FEd}(M_R) = n$. Then there exists an exact sequence of injective $R$-modules

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots,$$

where each $E^i$ is finitely cogenerated for any $i \geq n$. Since $S$ is projective, there is an exact sequence

$$0 \to \text{Hom}_R(S,M) \to \text{Hom}_R(S,E^0) \to \cdots \to \text{Hom}_R(S,E^n) \to \cdots$$

of injective $S$-modules $\text{Hom}_R(S,E^i)$, where by [13, Proposition 8.3], $\text{Hom}_R(S,E^i)$ is finitely cogenerated for any $i \geq n$. Thus $\text{FEd}(\text{Hom}_R(S,M))_s \leq n$ and hence, we have $\text{FEd}(\text{Hom}_R(S,M)) \leq \text{FEd}(M_R)$. □

**Proposition 2.18.** Let $S \geq R$ be a finite normalizing extension, $S_R$ be Projective, and $S$ be $R$-projective. Then for each right $S$-module $M_S$, $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S,M))$.

**Proof.** By [12, Lemma 1.1], $M_S$ is isomorphic to a direct summand of $\text{Hom}_R(S,M)$. So, from Corollary 2.6, we deduce that $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S,M))$. □

**Proposition 2.19.** Let $S \geq R$ be an almost excellent extension. Then for each right $S$-module $M_S$, $\text{FEd}(M_R) \leq \text{FEd}(M_S)$.

**Proof.** Assume that $\text{FEd}(M_S) = n$. So, there exists an exact sequence of injective $S$-modules

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots,$$

where each $E^i$ is finitely cogenerated for any $i \geq n$. Thus by [18, Proposition 5.1], every $E^i$ is an injective $R$-module and also, it is a finitely cogenerated $R$-module for $i \geq n$ by [14, Theorem 5]. Therefore, it follows that $\text{FEd}(M_R) \leq \text{FEd}(M_S)$. □

**Corollary 2.20.** Let $S \geq R$ be an almost excellent extension. Then for each right $S$-module $M_S$, $\text{FEd}(M_R) = \text{FEd}(M_S) = \text{FEd}(\text{Hom}_R(S,M))$. 
Theorem 2.21. Assume that $S \geq R$ is a finite normalizing extension and $S_R$ is Projective. Then:

1. If $S$ is $R$-projective and $\text{FED}(S) < \infty$, then $\text{FED}(S) \leq \text{FED}(R)$.
2. If $\text{FED}(R) < \infty$, then $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$, where $k = \text{id}(S_R)$ and $d = \sup\{\text{FED}(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FED}(M_S) = 0\}$.

Proof. (1) Assume that $\text{FED}(S) = n$ and $\text{FED}(M_S) = n$ for a finitely cogenerated $S$-module $M$. Since $S_R$ is projective, by hypothesis and [12, Lemma 1.1], $M_S$ is isomorphic to a direct summand of $\text{Hom}_R(S,M)$ and hence we have:

$$0 \to K \to \text{Hom}_R(S,M) \to M_S \to 0.$$ 

By [14, Lemma 4], $\text{Hom}_R(S,M)$ is finitely cogenerated $S$-module, since $M_R$ is a finitely cogenerated $R$-module. So, $\text{FED}(\text{Hom}_R(S,M)_S) \leq n$. On the other hand, by Theorem 2.5,

$$\text{FEd}(K) \leq \max\{n, n - 1\},$$

$$n = \text{FEd}(M_S) \leq \max\{\text{FEd}(\text{Hom}_R(S,M)_S), \text{FEd}(K_S) - 1\} \leq \text{FED}(S) = n.$$

Therefore $\text{FEd}(\text{Hom}_R(S,M)_S) = n$. Thus, Proposition 2.17 implies that

$$\text{FEd}(\text{Hom}_R(S,M)_S) \leq \text{FEd}(M_R)$$

and hence $\text{FED}(S) \leq \text{FED}(R)$.

(2) Assume that $\text{FED}(R) = n$ and $\text{FEd}(M_R) = n$ for a finitely cogenerated $R$-module $M$. Since $S_R$ is projective, by [12, Lemma 1.1], $M_R$ is isomorphic to a direct summand of $\text{Hom}_R(S,M)$ which induces the following short exact sequence of $R$-modules:

$$0 \to K \to \text{Hom}_R(S_R,M) \to M_R \to 0.$$ 

It is clear that $\text{Hom}_R(S_R,M)$ is a finitely cogenerated $R$-module. Thus Theorem 2.5 implies that

$$n = \text{FEd}(M_R) \leq \max\{\text{FEd}(\text{Hom}_R(S_R,M)), \text{FEd}(K_R) - 1\} \leq \text{FED}(R) = n,$$

and hence $\text{FEd}(\text{Hom}_R(S_R,M)) = n$.

If $\text{FEd}(\text{Hom}_R(S,M)) = m \leq \text{FED}(S)$, then there is an injective resolution

$$0 \to \text{Hom}_R(S,M) \xrightarrow{f_0} E^0 \xrightarrow{f_1} E^1 \to \cdots \to E^{m-1} \xrightarrow{f_m} E^m \xrightarrow{f_{m+1}} \cdots$$

of $\text{Hom}_R(S,M)$, where every $E^i$ is a finitely cogenerated $S$-module for any $i \geq m$. Let $D^i = \text{coker}(f_i)$ for every $i \geq 0$. Thus, the following short exact sequences

$$0 \to \text{Hom}_R(S,M) \to E^0 \to D^0 \to 0,$$

$$0 \to D^{m-2} \to E^{m-1} \to D^{m-1} \to 0,$$

$$0 \to D^{m-1} \to E^m \to D^m \to 0.$$
exists, where $\text{FE}(D^{m-1}) = 0$. But by hypothesis and Proposition 2.2, we have:

$$\text{FE}(D^i)_R \leq \text{id}(D^i)_R + 1 \leq \text{id}(S)_R + 1 = k + 1,$$

therefore by Theorem 2.5, we deduce that:

$$\text{FE}(D^{m-1})_R \leq \max\{\text{FE}(E^{m-1})_R, \text{FE}(D^{m-1})_R + 1\} = 1 + \max\{k, d\},$$

$$\text{FE}(D^{m-2})_R \leq \max\{\text{FE}(D^{m-2})_R, \text{FE}(D^{m-2})_R + 1\} < 2 + \max\{k, d\},$$

$$\vdots$$

$$\text{FE}(D^0)_R \leq \max\{\text{FE}(E^1)_R, \text{FE}(D^1)_R + 1\} < m - 1 + \max\{k, d\},$$

$$n = \text{FE}(\text{Hom}_R(S, M))_R \leq \max\{\text{FE}(E^0)_R, \text{FE}(D^0)_R + 1\} < m + \max\{k, d\}.$$

Thus $\text{FE}(R) < m + \max\{k, d\} \leq \text{FE}(S) + \max\{k, d\}$ and so, the proof is complete. $\square$

**Corollary 2.22.** Let $S \geq R$ be an almost excellent extension. Then $\text{FE}(R) < \text{FE}(S) + \text{id}(S)_R$.

**Proof.** By Proposition 2.19 and Theorem 2.21, this is clear. $\square$

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**References**