Copresented Dimension of Modules

M. Amini*, F. Hassani

Department of Mathematics, Payame Noor University, Tehran, Iran.

E-mail: mostafa.amini@pnu.ac.ir
E-mail: Hassani@pnu.ac.ir

Abstract. In this paper, a new homological dimension of modules, copresented dimension, is defined. We study some basic properties of this homological dimension. Some ring extensions are considered, too. For instance, we prove that if $S \geq R$ is a finite normalizing extension and $S_R$ is a projective module, then for each right $S$-module $M_S$, the copresented dimension of $M_S$ does not exceed the copresented dimension of $\text{Hom}_R(S,M)$.

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1. Introduction

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. First we recall some known notions and facts needed in the sequel. Let $R$ be a ring, $n$ a non-negative integer and $M$ an $R$-module. Then

1. $M$ is said to be finitely cogenerated [1] if for every family $\{V_k\}_J$ of submodules of $M$ with $\bigcap_j V_k = 0$, there is a finite subset $I \subset J$ with $\bigcap_j V_k = 0$.

2. $M$ is said to be $n$-copresented [14] if there is an exact sequence of $R$-modules $0 \to M \to E^0 \to E^1 \to \cdots \to E^n$, where each $E^i$ is a finitely cogenerated injective module.

*Corresponding Author

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(3) $R$ is called right co-coherent [17] if every finitely cogenerated factor module of a finitely cogenerated injective $R$-module is finitely copresented.

(4) $R$ is called $n$-co-coherent [14] in case every $n$-copresented $R$-module is $(n + 1)$-copresented. It is easy to see that $R$ is cocoherent if and only if it is 1-cocoherent. Recall that a ring $R$ is called right conoethrian [4] if every factor module of a finitely cogenerated $R$-module is finitely cogenerated. By [4, Proposition 17], a ring $R$ is co-noethrian if and only if it is 0-cocoherent.

(5) $M$ is said to be $n$-presented [5] if there is an exact sequence of $R$-modules $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$, where each $F_i$ is a finitely generated free module.

(6) $R$ is called coherent [18] in case every 0-presented $R$-module is 1-presented.

(7) A ring extension $R \subseteq R'$ with characteristic $p > 0$ is called a purely inseparable extension [10] if for every element $r' \in R'$, there exists a non-negative integer $n$ such that $r'^p \in R$.

(8) For any commutative ring $R$ of prime characteristic $p > 0$, assume that $F_R : R \rightarrow R^{(e)}$ is the $e$-th iterated Frobenius map in which $R^{(e)} \cong R$. Then, the perfect closure [9] of $R$, denoted by $R^\infty$, is defined as the limit of the following direct system:

\[ R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots \]

(9) $M$ is called $(n,d)$-injective [18] if $\text{Ext}_R^{d+1}(N,M) = 0$ for any $n$-presented right $R$-module $N$. It is clear that $M$ is $(0,0)$-injective if and only if $M$ is injective.

(10) Assume that $S \supseteq R$ is a unitary ring extension. Then, the ring $S$ is called right $R$-projective [6] in case, for any right $S$-module $M_S$ with an $S$-module $N_S$, $N_R | M_R$ implies $N_S | M_S$, where $N | M$ means that $N$ is a direct summand of $M$.

(11) The ring extension $S \supseteq R$ is called a finite normalizing extension [8] in case there is a finite subset $\{s_1, \ldots, s_n\} \subseteq S$ such that $S = \sum_{i=1}^n s_i R$ and $s_i R = Rs_i$ for $i = 1, \ldots, n$.

(12) A finite normalizing extension $S \supseteq R$ is called an almost excellent extension [12] in case $R S$ is flat, $S_R$ is projective, and the ring $S$ is right $R$-projective.

In this paper, we introduce the dual concepts of presented dimensions of $R$-modules. We also, introduce the copresented dimension of any $R$-module $M$:

\[ \text{Fed}(M) = \inf \{m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^m \rightarrow \cdots \rightarrow E^{m+i} \rightarrow \cdots, \text{ such that } E^{m+i} \text{ are finitely cogenerated for} \]
If \( K = \ker(E^m \rightarrow E^{m+1}) \), then \( K \) has an infinite finite copresentation. It is clear that any copresented dimension is finitely copresented dimension (see [16]). Also, the copresented dimension of ring \( R \) is defined to be:

\[
\text{FED}(R) = \sup \{ \text{FEd}(M) \mid M \text{ is a finitely cogenerated module} \}.
\]

Then, some basic properties of the copresented dimensions of modules are studied. For example, it is shown that if \( \text{FEd}(M) < \infty \), then \( \text{id}(M) \leq n \) if and only if \( \text{Ext}_R^{n+1}(N, M) = 0 \) for every strongly copresented \( R \)-module \( N \). Also, it is proved that \( \text{FED}(R \oplus S) = \sup \{ \text{FED}(R), \text{FED}(S) \} \), for any two rings \( R \) and \( S \). Also, some characterizations of the copresented dimensions of modules on Ring Extensions are determined. For instance, let \( S \geq R \) be a finite normalizing extension with \( S \) projective as an \( R \)-module, then for any right \( R \)-module \( M \), we have \( \text{FEd}(\text{Hom}_R(S, M)) = \text{FEd}(M_R) \). Finally, we give a sufficient condition under which \( \text{FED}(S) \leq \text{FED}(R) \) and or \( \text{FED}(R) < \text{FED}(S) + \max\{k, d\} \), where \( k = \text{id}(S_R) \) and \( d = \sup \{ \text{FEd}(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FEd}(M_S) = 0 \} \).

2. Main Results

We start this section with the following definition which is the dual of the presented dimension of a module.

**Definition 2.1.** For any \( R \)-module \( M \), we define the copresented dimension of \( M \) to be \( \text{FEd}(M) = \inf \{ m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^m \rightarrow \cdots \rightarrow E^{m+i} \rightarrow \cdots, \text{ so that } E^{m+i} \text{ are finitely cogenerated for } i = 0, 1, 2, \cdots \} \). In particular, a module \( M \) is called strongly copresented module if \( \text{FEd}(M) = 0 \).

**Proposition 2.2.** For any \( R \)-module \( M \), \( \text{FEd}(M) \leq \text{id}(M) + 1 \).

**Proof.** It is a direct consequence of Definition 2.1. \( \square \)

**Example 2.3.** Let \( R = \mathbb{Z} \). Since \( \text{id}(\mathbb{Z}_{p^\infty}) = 0 \), we have \( \text{FEd}(\mathbb{Z}_{p^\infty}) \leq 1 \). On the other hand, \( \mathbb{Z}_{p^\infty} \) is finitely cogenerated by [1, p.124]. So by Definition 2.1, \( \text{FEd}(\mathbb{Z}_{p^\infty}) = 0 \).

Now, we study the behavior of the copresented dimension on the exact sequences. Before this we need the following lemma.

**Lemma 2.4.** Let \( 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \) be a short exact sequence of \( R \)-modules. Then:

1. If \( 0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \) and \( 0 \rightarrow C \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \) are injective resolutions of \( A \) and \( C \), respectively. Then the exact sequence

\[
0 \rightarrow B \rightarrow A^0 \oplus C^0 \rightarrow A^1 \oplus C^1 \rightarrow \cdots
\]

is an injective resolution of \( B \).
If \( 0 \rightarrow B \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \) and \( 0 \rightarrow C \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots \) are injective resolutions of \( B \) and \( C \), respectively. Then the exact sequence
\[
0 \rightarrow A \rightarrow B_0 \rightarrow D_0 \rightarrow D_1 \rightarrow \cdots
\]
is an injective resolution of \( A \), where \( D_i = C_i \oplus B^{i+1} \) for any \( i \geq 0 \).

If \( 0 \rightarrow B \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \) and \( 0 \rightarrow A \rightarrow A_0 \rightarrow A_1 \rightarrow \cdots \) are injective resolutions of \( B \) and \( A \), respectively. Then the exact sequence
\[
0 \rightarrow C \rightarrow F_0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots
\]
is an injective resolution of \( C \), where \( F_0 = B_0 \oplus A_1 \) and \( E_i = A_0 \oplus B^{i+1} \oplus A^{i+2} \) for any \( i \geq 0 \).

Proof. (1) The proof is similar to that of [3, Theorem 2.4].

(2) Let \( 0 \rightarrow B \rightarrow B_0 \rightarrow B_1 \rightarrow \cdots \) be an injective resolution of \( B \). Then, the exact sequences
\[
0 \rightarrow K \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \quad \text{and} \quad 0 \rightarrow B \rightarrow B_0 \rightarrow K \rightarrow 0
\]
exist, where \( K = B_0^0 \). Now, we consider the following commutative diagram:
\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & A & B & C & 0 \\
\| & \downarrow & \downarrow \\
0 & A & B_0 & D & 0 \\
\downarrow & \downarrow \\
K & = & K \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

By (1), there is an exact sequence
\[
0 \rightarrow D \rightarrow D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \cdots
\]
of injective \( R \)-modules \( D^i \) such that \( D^i = C^i \oplus B^{i+1} \) for any \( i \geq 0 \).

Combining this sequence with the exact sequence \( 0 \rightarrow A \rightarrow B_0 \rightarrow D \rightarrow 0 \), we get the exact sequence
\[
0 \rightarrow A \rightarrow B^0 \rightarrow D^0 \rightarrow D^1 \rightarrow \cdots,
\]
where \( B^0 \) and \( D^i \) are injective for any \( i \geq 0 \).

(3) Let \( 0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots \) be an injective resolution of \( A \). Then, the exact sequences
\[
0 \rightarrow K \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots \quad \text{and} \quad 0 \rightarrow A \rightarrow A^0 \rightarrow K \rightarrow 0
\]
exist, where \( K = A^0_A \). Now, we consider the following commutative diagram:
By (1), there is an exact sequence
\[ 0 \rightarrow F \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots \]
of injective \( R \)-modules \( F^i \) such that \( F^i = B^i \oplus A^{i+1} \) for any \( i \geq 0 \).

It is clear that \( F = A^0 \oplus C \). So, the exact sequence \( 0 \rightarrow C \rightarrow F \rightarrow A^0 \rightarrow 0 \)
exists. Let \( K = \frac{F^0}{F} \), then we obtain the following commutative diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow & C \\
\| & & \downarrow \\
0 & \rightarrow & C \\
\downarrow & & \downarrow \\
0 & \rightarrow & F \\
\downarrow & & \downarrow \\
0 & \rightarrow & E \\
\downarrow & & \downarrow \\
K & = & K \\
\downarrow & & \downarrow \\
0 & 0 & 0
\end{array}
\]

Therefore by (1), the sequence
\[ 0 \rightarrow E \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots \]
is an injective resolution of \( E \), where \( E^i = A^0 \oplus F^{i+1} = A^0 \oplus B^{i+1} \oplus A^{i+2} \) for any \( i \geq 0 \). Combining this sequence with the exact sequence \( 0 \rightarrow C \rightarrow F^0 \rightarrow E \rightarrow 0 \), we get the exact sequence
\[ 0 \rightarrow C \rightarrow F^0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots , \]
where \( F^0 \) and \( E^i \) are injective for any \( i \geq 0 \). □

**Theorem 2.5.** Let \( 0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0 \) be an exact sequence of \( R \)-modules. Then \( \text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\} \), \( \text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\} \), \( \text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\} \).

**Proof.** Assume that \( E' \) is an injective resolution of \( A \) and \( E'' \) is an injective resolution of \( C \). Thus by Lemma 2.5(1), there exists an injective resolution \( E \) of \( B \) such that
\[ 0 \rightarrow E' \rightarrow E^B = E' \oplus E'' \rightarrow E'' \rightarrow 0 \]
is an exact sequence of complexes. Hence for every \( m \geq \max\{\text{FEd}(A), \text{FEd}(C)\} \), \( E^m \) is finitely cogenerated. So, we deduce that \( \text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\} \).

Assume that \( E \) is an injective resolution of \( C \) and \( E \) is an injective resolution of \( B \). Thus by Lemma 2.5(2), the exact sequence

\[
0 \to A \to E^0 \to D^0 \to D^1 \to \cdots \to D^d \to \cdots
\]

is an injective resolution of \( A \). So for every \( d \geq \max\{\text{FEd}(B), \text{FEd}(C) - 1\} \), \( D^d \) is finitely cogenerated. Thus, we have that \( \text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\} \). Also, it is prove that \( \text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\} \). \( \square \)

The proof of the following Corollary is similar to the proof of [19, Corollary 2.7].

**Corollary 2.6.** If \( \text{FEd}(M_1), \text{FEd}(M_2), \cdots, \text{FEd}(M_d) \) are finite, then:

\[
\text{FEd}(\oplus M_i) = \max\{\text{FEd}(M_i) \mid i = 1, \cdots, d\}.
\]

**Proof.** For the case \( m = 2 \), the exact sequences

\[
0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0
\]

and

\[
0 \to M_2 \to M_2 \oplus M_1 \to M_1 \to 0
\]

exist. Thus by Theorem 2.5, we deduce that

\[
\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_1) - 1\},
\]

\[
\text{FEd}(M_1) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 1\}
\]

and

\[
\text{FEd}(M_1 \oplus M_2) \leq \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}.
\]

Assume that \( \text{FEd}(M_1) < \text{FEd}(M_2) \). Then \( \text{FEd}(M_1) \leq \text{FEd}(M_2) - 1 \), and we have:

\[
\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 2\} = \text{FEd}(M_1 \oplus M_2).
\]

Also, similarly \( \text{FEd}(M_1) \leq \text{FEd}(M_1 \oplus M_2) \). So, we conclude that \( \text{FEd}(M_1 \oplus M_2) = \max\{\text{FEd}(M_1), \text{FEd}(M_2)\} \). \( \square \)

**Proposition 2.7.** Let \( n \) be a non-negative integer. Then the following statements are equivalent:

1. \( \text{id}(M) \leq n \) for every strongly copresented \( R \)-module \( M \);

2. \( \text{Ext}^{n+1}_{R}(N, M) = 0 \) for every strongly copresented \( R \)-module \( N \).

**Proof.** (1) \( \Rightarrow \) (2) This is obvious.

(2) \( \Rightarrow \) (1) We use the induction on \( n \). Let \( n = 0 \). Since \( \text{Ext}^1_R(N, M) = 0 \) for any strongly copresented \( R \)-module \( N \), by using the exact sequence \( 0 \to M \to E^0 \to L^0 \to 0 \) where \( E^0 \) is finitely cogenerated and \( L^0 \) is strongly copresented, we deduce that \( \text{Ext}^1_L(L^0, M) = 0 \). Therefore by [7, Theorem 7.31], the exact sequence above is split. So, \( M \) is injective and hence \( \text{id}(M) \leq 0 \). Assume that
n > 0. By [7, Corollary 6.42], we have that \( \text{Ext}^{n+1}_R(N, M) \cong \text{Ext}^n_R(N, L^0) = 0 \). Thus by induction hypothesis, \( \text{id}(L^0) \leq n - 1 \). Therefore from the exact sequence above, we deduce that \( \text{id}(M) \leq n \).

**Proposition 2.8.** Let \( \text{FEd}(M) \leq 1 \). Then the following statements are equivalent:

1. \( \text{id}(M) \leq n; \)
2. \( \text{Ext}^{n+1}_R(N, M) = 0 \) for every strongly copresented \( R \)-module \( N \).

**Proof.** Since \( \text{FEd}(M) \leq 1 \), the exact sequence \( 0 \to M \to E^0 \to L^0 \to 0 \) exists, where \( E^0 \) is injective and \( L^0 \) is strongly copresented. Thus, \( \text{Ext}^{n+1}_R(N, M) = 0 \) for any strongly copresented \( R \)-module \( N \) if and only if \( \text{Ext}^n_R(N, L^0) = 0 \) if and only if \( \text{id}(L^0) \leq n - 1 \) (by Proposition 2.7) if and only if \( \text{id}(M) \leq n \). □

**Theorem 2.9.** Let \( \text{FEd}(M) < \infty \). Then the following statements are equivalent:

1. \( \text{id}(M) \leq n; \)
2. \( \text{Ext}^{n+1}_R(N, M) = 0 \) for every strongly copresented \( R \)-module \( N \).

**Proof.** (1) \( \Rightarrow \) (2) It is clear.

(2) \( \Rightarrow \) (1) If \( \text{FEd}(M) = m \), then the exact sequence

\[
0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \xrightarrow{d^{m-1}} E^m \xrightarrow{d^m} \cdots \to E^{m+j} \to \cdots
\]

exists, where \( E^i \) is finitely cogenerated for any \( i \geq m \). By Proposition 2.2, \( n + 1 \geq m \). Let \( \text{Ext}^{n+1}_R(N, M) = 0 \) for every strongly copresented \( R \)-module \( N \). Thus by [7, Corollary 6.42], we have

\[
\text{Ext}^{n+1}_R(N, M) \cong \text{Ext}^{n-m+1}_R(N, \text{coker}d^{m-1}) = 0.
\]

Since \( \text{coker}d^{m-1} \) is strongly copresented, Proposition 2.8 implies that

\[
\text{id}(\text{coker}d^{m-1}) \leq n - m
\]

and so, we deduce that \( \text{id}(M) \leq n \). □

**Corollary 2.10.** Let \( D(R) < \infty \). Then:

\[
D(R) = \sup\{\text{pd}(N) \mid N \text{ is strongly copresented}\}.
\]

**Proof.** Assume that \( D(R) \leq m \). Thus, \( \text{pd}(N') \leq m \) for any \( R \)-module \( N' \). So, for any strongly copresented \( R \)-module \( N \), \( \text{pd}(N) \leq m \). Conversely, let \( \text{pd}(N) \leq m \) for every strongly copresented \( R \)-module \( N \). Thus \( \text{Ext}^{m+1}_R(N, M) = 0 \) for every strongly presented \( R \)-module \( M \). Since \( D(R) < \infty \), \( \text{FEd}(M) < \infty \) by Proposition 2.2. Therefore by Theorem 2.9, \( \text{id}(M) \leq m \) and hence by [19, corollary 3.7], \( D(R) \leq m \). □

**Definition 2.11.** For any ring \( R \), we define the copresented dimension of \( R \) to be \( \text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\} \).
Example 2.12. Let $R = k[x^3, x^3y, xy^3, y^3]$, where $k$ is a field with characteristic $p = 3$. By Definition 2.11 and Proposition 2.2, $\text{FED}(R^\infty) \leq D(R^\infty) + 1$, where $R^\infty$ is perfect closure of $R$. On the other hand, $k[x, y]$ is purely inseparable over $R$. Also, by [9, Proposition 3.3], $(k[x, y])^\infty$ is coherent. Therefore by [10, Remark 1.4], $R^\infty$ is coherent. Since $R$ is reduced, [2, Proposition 5.5] implies that $\text{FED}(R^\infty) \leq \dim(R) + 1$ and so, $\text{FED}(R^\infty) \leq 3$.

Proposition 2.13. The following statements are equivalent:

1. $\text{FED}(R) = 0$;
2. Every finitely cogenerated module has an infinite finite copresentation;
3. Every finitely cogenerated module is finitely copresented;
4. $R$ is co-Noetherian.

Proof. The implication (1) $\implies$ (2) $\implies$ (3) follow immediately from Definition 2.11. (3) $\implies$ (4) $\implies$ (1) are trivial. □

Corollary 2.14. If $\text{FED}(R) \leq 0$, then $R$ is $n$-coherent.

Proof. Since every $n$-copresented module $M$ is finitely cogenerated, Proposition 2.13 implies that $M$ is $(n + 1)$-copresented. □

Next, we study the copresented dimension of the direct sum of rings. But before this we need the following lemma.

Lemma 2.15. Let $f : R \to S$ be a ring epimorphism. If $M_S$ is a right $S$-module (hence a right $R$-module) and $N_R$ is a right $R$-module, then the following statements hold:

1. $M \otimes_R S \cong M_S$,
2. If $f$ is flat and $N_R$ is a finitely cogenerated right $R$-module, then $N \otimes_R S$ is a finitely cogenerated right $S$-module.
3. If $f$ is flat, then $M_S$ is a finitely cogenerated right $S$-module if and only if $M_R$ is a finitely cogenerated right $R$-module.
4. If $f$ is projective, then $M_S$ is an injective right $S$-module if and only if $M_R$ is an injective right $R$-module.

Proof. (1) This is clear.

(2) For any family of submodules $\{N_i \otimes_R 1_S | i \in I\}$ in $N \otimes_R S$, if $\bigcap_{i \in I} (N_i \otimes_R 1_S) = 0$, then we need to show that $\bigcap_{i \in I} (N_i \otimes_R 1_S) = 0$ for some finite subset $F$ of $I$. Since $f$ is flat, we have that $\bigcap_{i \in I} N_i \otimes_R 1_S = 0$. So, $\bigcap_{i \in F} N_i = 0$ and hence by hypotises $\bigcap_{i \in F} N_i = 0$ for some finite subset $F$ of $I$. Therefore, $\bigcap_{i \in F} (N_i \otimes_R 1_S) = \bigcap_{i \in F} N_i \otimes_R 1_S = 0$.

(3) $(\Rightarrow)$: Let $\psi : M \to \prod_{i \in I} R$ is a monomorphism, then we claim that $\pi : M \to \prod_{i \in F} R$ is a monomorphism for some finite subset $F$ of $I$. We have the following commutative diagram:
\[ M \xrightarrow{\psi} \prod_{i \in I} R \xrightarrow{\sim} \prod_{i \in I} S, \]

where since \( g \) is epimorphism and \( \psi \) is monomorphism, \( h \) is monomorphism. So by hypothesis, \( \alpha : M \to \prod_{i \in F} S \) is a monomorphism for some finite subset \( F \) of \( I \). Therefore the following commutative diagram:

\[ M \xrightarrow{\gamma} \prod_{i \in F} R \xrightarrow{\sim} \prod_{i \in F} S, \]

where \( \beta \) is epimorphism and \( \alpha \) is monomorphism, implies that \( \gamma \) is monomorphism.

(\( \Leftarrow \)) : This follows from (1) and (2)

(4) By [5, Lemma 3.3], \( M_S \) is an \((n, d)\)-injective right \( S \)-module if and only if \( M_R \) is an \((n, d)\)-injective right \( R \)-module. If \( n = 0, d = 0 \), then (4) is hold. \( \square \)

**Theorem 2.16.** Assume that \( R \) and \( S \) are two rings. Then:

\[ \text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}. \]

**Proof.** We first show that \( \text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\} \). Consider \( \text{FED}(R) = n, \text{FED}(S) = m \) and \( n \geq m \). Also, let \( M \) be a finitely cogenerated right \((R \oplus S)\)-module. Then \( M \) has a unique decomposition \( M = A \oplus B \), where \( A, B \) are right modules of rings \( R \) and \( S \), respectively. By [15, Lemma 1.1], \( A \) and \( B \) are finitely cogenerated right \((R \oplus S)\)-module. So by Lemma 2.15, \( A \) is finitely cogenerated right \( R \)-module and \( B \) is finitely cogenerated right \( S \)-module. Therefore \( \text{Fed}(A) \leq n \) and \( \text{Fed}(B) \leq m \), and hence there is an exact sequences

\[ 0 \to A \to E^0_a \to E^1_a \to \cdots \to E^{n-1}_a \to E^n_a \to \cdots, \]

\[ 0 \to B \to E^0_b \to E^1_b \to \cdots \to E^{m-1}_b \to E^m_b \to \cdots \]

of injective right \( R \)-modules \( E^i_a \) and injective right \( S \)-modules \( E^i_b \) such that \( E^i_a, E^i_b \) are finitely cogenerated for any \( i \geq n \) and \( i \geq m \), respectively. So, we deduce that the exact sequence

\[ 0 \to A \oplus B \to E^0_a \oplus E^0_b \to E^1_a \oplus E^1_b \to \cdots \to E^{n-1}_a \oplus E^{m-1}_b \to E^n_a \oplus E^m_b \to \cdots \]

exists, where by Lemma 2.15, every \( E^i_a \oplus E^i_b \) is injective right \((R \oplus S)\)-module and also, every \( E^i_a \oplus E^i_b \) is finitely cogenerated for any \( i \geq n \). Therefore, we have \( \text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\} \).

Conversely, Assume that \( \text{FED}(R \oplus S) = d \). If \( M \) is a finitely cogenerated right \( R \)-module. Then by Lemma 2.15, \( M \) is a finitely cogenerated right \((R \oplus S)\)-module and hence \( \text{FED}(M_{(R \oplus S)}) \leq d \). Thus, the exact sequence \( 0 \to M \to E^0 \to E^1 \to \cdots \to E^{d-1} \to E^d \to \cdots \) of injective right \((R \oplus S)\)-modules

$E^i$ exists, where every $E^i$ is finitely cogenerated for any $i \geq d$. Let $E^i = C^i \oplus D^i$, where $C^i$ is a $R$-module and $D^i$ is a $S$-module. On the other hand, $M$ is a right $R$-module, so we have the exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^{d-1} \to C^d \to \cdots$ of $R$-modules. But, every $C^i$ is injective right $(R \oplus S)$-module and also every $C^i$ is finitely cogenerated right $(R \oplus S)$-module for $i \geq d$. So by [15, Lemma 1.1] and Lemma 2.15, $C^i$ is an injective right $R$-module and it is finitely cogenerated $R$-module for $i \geq d$. Therefore $\text{FEd}(M) \leq d$ and hence $\text{FED}(R) \leq d$. Similarly, $\text{FED}(S) \leq d$ and implies that $\text{sup}\{\text{FED}(R), \text{FED}(S)\} \leq \text{FED}(R \oplus S)$. □

**Proposition 2.17.** Let $S \geq R$ be a finite normalizing extension with $S_R$ projective as an $R$-module. Then for any right $R$-module $M_R$, $\text{FEd}(\text{Hom}_R(S,M ))_S \leq \text{FEd}(M_R)$.

**Proof.** Assume that $\text{FEd}(M_R) = n$. Then there exists an exact sequence of injective $R$-modules

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots,$$

where each $E^i$ is finitely cogenerated for any $i \geq n$. Since $S$ is projective, there is an exact sequence

$$0 \to \text{Hom}_R(S,M) \to \text{Hom}_R(S,E^0) \to \cdots \to \text{Hom}_R(S,E^n) \to \cdots$$

of injective $S$-modules $\text{Hom}_R(S,E^i)$, where by [13, Proposition 8.3], $\text{Hom}_R(S,E^i)$ is finitely cogenerated for any $i \geq n$. Thus $\text{FEd}(\text{Hom}_R(S,M ))_S \leq n$ and hence, we have $\text{FEd}(\text{Hom}_R(S,M))_S \leq \text{FEd}(M_R)$. □

**Proposition 2.18.** Let $S \geq R$ be a finite normalizing extension, $S_R$ be Projective, and $S$ be $R$-projective. Then for each right $S$-module $M_S$, $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S,M ))$.

**Proof.** By [12, Lemma 1.1], $M_S$ is isomorphic to a direct summand of $\text{Hom}_R(S,M)$. So, from Corollary 2.6, we deduce that $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S,M))$. □

**Proposition 2.19.** Let $S \geq R$ be an almost excellent extension. Then for each right $S$-module $M_S$, $\text{FEd}(M_R) \leq \text{FEd}(M_S)$.

**Proof.** Assume that $\text{FEd}(M_S) = n$. So, there exists an exact sequence of injective $S$-modules

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots,$$

where each $E^i$ is finitely cogenerated for any $i \geq n$. Thus by [18, Proposition 5.1], every $E^i$ is an injective $R$-module and also, it is a finitely cogenerated $R$-module for $i \geq n$ by [14, Theorem 5]. Therefore, it follows that $\text{FEd}(M_R) \leq \text{FEd}(M_S)$. □

**Corollary 2.20.** Let $S \geq R$ be an almost excellent extension. Then for each right $S$-module $M_S$, $\text{FEd}(M_R) = \text{FEd}(M_S) = \text{FEd}(\text{Hom}_R(S,M ))$. 

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Theorem 2.21. Assume that $S \geq R$ is a finite normalizing extension and $S_R$ is Projective. Then:

1. If $S$ is $R$-projective and $\text{FED}(S) < \infty$, then $\text{FED}(S) \leq \text{FED}(R)$.
2. If $\text{FED}(R) < \infty$, then $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$, where $k = \text{id}(S_R)$ and $d = \sup\{\text{FED}(M_R) \mid M \in \text{Mod} - S$ and $\text{FED}(M_S) = 0\}$.

Proof. (1) Assume that $\text{FED}(S) = n$ and $\text{FED}(M_S) = n$ for a finitely cogenerated $S$-module $M$. Since $S_R$ is projective, by hypothesis and [12, Lemma 1.1], $M_S$ is isomorphic to a direct summand of $\text{Hom}_R(S, M)$ and hence we have:

$$0 \to K \to \text{Hom}_R(S, M) \to M_S \to 0.$$ 

By [14, Lemma 4], $\text{Hom}_R(S, M)$ is finitely cogenerated $S$-module, since $M_R$ is a finitely cogenerated $R$-module. So, $\text{FED}(\text{Hom}_R(S, M)_S) \leq n$. On the other hand, by Theorem 2.5,

$$\text{FED}(K) \leq \max\{n, n - 1\},$$

$$n = \text{FED}(M_S) \leq \max\{\text{FED}(\text{Hom}_R(S, M)_S), \text{FED}(K_S) - 1\} \leq \text{FED}(S) = n.$$ 

Therefore $\text{FED}(\text{Hom}_R(S, M)_S) = n$. Thus, Proposition 2.17 implies that

$$\text{FED}(\text{Hom}_R(S, M)_S) \leq \text{FED}(M_R)$$

and hence $\text{FED}(S) \leq \text{FED}(R)$.

(2) Assume that $\text{FED}(R) = n$ and $\text{FED}(M_R) = n$ for a finitely cogenerated $R$-module $M$. Since $S_R$ is projective, by [12, Lemma 1.1], $M_R$ is isomorphic to a direct summand of $\text{Hom}_R(S, M)$ which induces the following short exact sequence of $R$-modules:

$$0 \to K \to \text{Hom}_R(S_R, M) \to M_R \to 0.$$ 

It is clear that $\text{Hom}_R(S_R, M)$ is a finitely cogenerated $R$-module. Thus Theorem 2.5 implies that

$$n = \text{FED}(M_R) \leq \max\{\text{FED}(\text{Hom}_R(S_R, M)_S), \text{FED}(K_R) - 1\} \leq \text{FED}(R) = n,$$

and hence $\text{FED}(\text{Hom}_R(S_R, M)) = n$. If $\text{FED}(\text{Hom}_R(S_R, M)_S) = m \leq \text{FED}(S)$, then there is an injective resolution

$$0 \to \text{Hom}_R(S, M) \xrightarrow{f_0} E^0 \xrightarrow{f_1} E^1 \to \cdots \to E^{m-1} \xrightarrow{f_m} E^m \xrightarrow{f_{m+1}} \cdots$$

of $\text{Hom}_R(S, M)$, where every $E^i$ is a finitely cogenerated $S$-module for any $i \geq m$. Let $D^i = \text{coker}(f_i)$ for every $i \geq 0$. Thus, the following short exact sequences

$$0 \to \text{Hom}_R(S, M) \to E^0 \to D^0 \to 0,$$

$$\cdots$$

$$0 \to D^{m-2} \to E^{m-1} \to D^{m-1} \to 0,$$

$$0 \to D^{m-1} \to E^m \to D^m \to 0.$$
exists, where $\text{FEd}(D^{m-1}) = 0$. But by hypothesis and Proposition 2.2, we have:

$$\text{FEd}(D^i)_R \leq \text{id}(D^i)_R + 1 \leq \text{id}(S)_R + 1 = k + 1,$$

$$\text{FEd}(D^{m-1})_R \leq d.$$ Therefore by Theorem 2.5, we deduce that:

$$\text{FEd}(D^{m-2})_R \leq \max\{\text{FEd}(E^{m-1})_R, \text{FEd}(D^{m-1})_R + 1\} < \max\{k + 1, d + 1\},$$

$$\text{FEd}(D^{m-3})_R \leq \max\{\text{FEd}(E^{m-2})_R, \text{FEd}(D^{m-2})_R + 1\} < 2 + \max\{k, d\},$$

$$\vdots$$

$$\text{FEd}(D^0)_R \leq \max\{\text{FEd}(E^1)_R, \text{FEd}(D^1)_R + 1\} < m - 1 + \max\{k, d\},$$

$n = \text{FEd}((\text{Hom}_R(S, M))_R \leq \max\{\text{FEd}(E^0)_R, \text{FEd}(D^0)_R + 1\} < m + \max\{k, d\}.$

Thus $\text{FED}(R) < m + \max\{k, d\} \leq \text{FED}(S) + \max\{k, d\}$ and so, the proof is complete. \hfill $\Box$

**Corollary 2.22.** Let $S \geq R$ be an almost excellent extension. Then $\text{FED}(R) < \text{FED}(S) + \text{id}(S)_R$.

**Proof.** By Proposition 2.19 and Theorem 2.21, this is clear. \hfill $\Box$

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**References**