Approximation by \((p, q)\)-Lupa\c{s} Stancu Operators

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Abstract. In this paper, \((p, q)\)-Lupa\c{s} Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of \((p, q)\)-Lupa\c{s} Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: \((p, q)\)-Integers, Lupa\c{s} \((p, q)\)-Bernstein Stancu operators, Statistical approximation, Korovkin’s type approximation.


1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators \(B_n : C[0, 1] \rightarrow C[0, 1]\) defined for any \(n \in \mathbb{N}\) and for any function \(f \in C[0, 1]\)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1].
\]

(1.1)

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].

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Further, based on $q$-integers, Lupaş [21] introduced the first $q$-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another $q$-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on $q$-integers have been introduced and their approximation properties studied.

Recently, the applications of $(p, q)$-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving $(p, q)$-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen et al [27] introduced the concept of post quantum calculus in approximation theory and constructed the $(p, q)$-analogue of Bernstein operators defined as follows for $0 < q < p \leq 1$:

$$B_{n,p,q}(f; x) = \frac{1}{p^\binom{n-1}{2}} \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{k}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^n - [n]_{p,q}}\right), \quad x \in [0, 1].$$ (1.2)

Note when $p = 1$, $(p, q)$-Bernstein Operators given by (1.2) turns out to be Phillips $q$-Bernstein Operators [38].

Also, we have

$$(1 - x)^n_{p,q} = \prod_{s=0}^{n-1} (p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2 x)...(p^{n-1} - q^{n-1} x)
= \sum_{k=0}^{n} (-1)^k p^{\binom{n-k}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} x^k.$$

Further, they applied the concept of $(p, q)$-calculus in approximation theory and studied approximation properties based on $(p, q)$-integers for Bernstein-Stancu operators, $(p, q)$-analogue of Bernstein-Kantorovich, $(p, q)$-analogue of Bernstein-Shurer operators, $(p, q)$-analogue of Bleimann-Butzer-Hahn operators and $(p, q)$-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined $(p, q)$-analogue of Lupaş Bernstein operators [17] as follows:
For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0,1] \to C[0,1]$ as

$$L^n_{p,q}(f; x) = \sum_{k=0}^{n} \frac{f\left(\frac{p^{n-k}}{\binom{n}{k}_{p,q}}\right)}{\prod_{j=1}^{n}(p^{-1}(1-x) + q^{j-1}x)^{n-k}} p^{\frac{(n-k)(q-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k},$$

(1.3)

are $(p, q)$-analogue of Lupas Bernstein operators.

Again when $p = 1$, Lupas $(p, q)$-Bernstein operators turns out to be Lupas $q$-Bernstein operators as given in [22, 37].

When $p = q = 1$, Lupas $(p, q)$-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau’s algorithm and Korovkin’s type approximation properties [17]: de-Casteljau’s algorithm and related results of degree elevation reduction for Bézier curves and surfaces holds for all $p > 0$ and $q > 0$. However to study Korovkin’s type approximation properties for Lupas $(p, q)$-analogue of the Bernstein operators, $0 < q < p \leq 1$ is needed.

Based on Korovkin’s type approximation, they proved that the sequence of $(p, q)$-analogue of Lupas Bernstein operators $L^n_{p,q}(f, x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $0 < q_n < p_n \leq 1$ such that $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} p_n = 1$ and $\lim_{n \to \infty} q^n_n = 1$. On the other hand, for any $p > 0$ fixed and $p \neq 1$, the sequence $L^n_{p,q}(f, x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Furthermore, in comparison to $q$-Bézier curves and surfaces based on Lupas $q$-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter $p$ have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of $(p, q)$-calculus.

For any $p > 0$ and $q > 0$, the $(p, q)$ integers $[n]_{p,q}$ are defined by
\[ [n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q}, & \text{when } p \neq q \neq 1 \\ n \ p^{n-1}, & \text{when } p = q \neq 1 \\ [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases} \]

where \([n]_q\) denotes the \(q\)-integers and \(n = 0, 1, 2, \ldots\).

The formula for \((p,q)\)-binomial expansion is as follows:

\[
(ax + by)_{p,q}^n := \sum_{k=0}^{n} \binom{n}{k}_{p,q} \frac{(n-k)(n-k-1)}{k(k-1)} a^{n-k} b^k x^k y^k,
\]

\[
(x + y)_{p,q}^n = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),
\]

\[
(1 - x)_{p,q}^n = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x),
\]

where \((p,q)\)-binomial coefficients are defined by

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.
\]

Details on \((p,q)\)-calculus can be found in [10, 11, 27].

Also, we have \((p,q)\)-analogue of Euler’s identity as:

\[
(1 - x)_{p,q}^n = \prod_{k=0}^{n-1} (p^k - q^k x) = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x)
\]

\[
= \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} x^k.
\]

Again by some simple calculations and using the property of \((p,q)\)-integers, we get \((p,q)\)-analogue of Pascal’s relation as follows:

\[
\binom{n}{k}_{p,q} = q^{n-k} \binom{n-1}{k-1}_{p,q} + p^k \binom{n-1}{k}_{p,q}, \quad (1.4)
\]

\[
\binom{n}{k}_{p,q} = p^{n-k} \binom{n-1}{k-1}_{p,q} + q^k \binom{n-1}{k}_{p,q}. \quad (1.5)
\]

We recall some results from [17] for Lupas \((p,q)\)-Bernstein operators, which reproduces linear and constant functions.
Some auxiliary results:

(1) \( L^n_{p,q}(1, \frac{u}{u+1}) = 1 \)

(2) \( L^n_{p,q}(t, \frac{u}{u+1}) = \frac{u}{u+1} \)

(3) \( L^n_{p,q}(t^2, \frac{u}{u+1}) = \frac{u}{u+1} p^{n-1} + \frac{qu}{u+1} \frac{[n-1]_{p,q}}{[n]_{p,q}} \)

or equivalently for \( x = \frac{u}{u+1} \)

\[
L^n_{p,q}(1, x) = 1, \\
L^n_{p,q}(t, x) = x, \\
L^n_{p,q}(t^2, x) = p^{n-1} + q^2 x^2 + \frac{[n-1]_{p,q}}{[n]_{p,q}} \]

2. Construction of \((p, q)\)-Lupaš Stancu Operators

In this section, we introduce \((p, q)\)-Lupaš Stancu operators as follows:

For any \( p > 0 \) and \( q > 0 \), the linear operators \( L^n_{p,q} : C[0, 1] \to C[0, 1] \)

\[
L^{\alpha,\beta}_{n,p,q}(f; x) = \sum_{k=0}^{n} f \left( \binom{n}{k}_{p,q} p^{n-k} + \frac{\alpha}{[n]_{p,q}} \right) b^{k,n}_{p,q}(t) 
\]

and \( b^{k,n}_{p,q}(t) \) is given by

\[
b^{k,n}_{p,q}(t) = \frac{\binom{n}{k}_{p,q} p^{(n-k)(n-k-1)} q^{k(k-1)} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}} 
\]

where \( 0 < \alpha < \beta \).

We give some equalities for operators (2.1) in the following lemma.

**Lemma 4.1.** The following equalities are true:

(i) \( L^{\alpha,\beta}_{n,p,q}(1; x) = 1 \),

(ii) \( L^{\alpha,\beta}_{n,p,q}(t; x) = \frac{[n]_{p,q} x^{n-x+x+\alpha}}{[n]_{p,q} + \beta} \),

(iii) \( L^{\alpha,\beta}_{n,p,q}(t^2; x) = \frac{1}{([n]_{p,q} + \beta)^2} \left( \frac{q^2}{p^{1-x} + q^2} x^2 + \frac{[n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2} x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right) 
\)
Proof. Proof of part (i) is obvious.

\[
L^\alpha,\beta_{n,p,q}(t; x) = \sum_{k=0}^{n} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b^k_{n,p,q}(t)
\]

\[
= \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L^n_{p,q}(t; x) + \frac{[\alpha]}{[n]_{p,q} + [\beta]} L^n_{p,q}(1; x).
\]

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

\[
L^\alpha,\beta_{n,p,q}(t^2; x) = \sum_{k=0}^{n} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b^k_{n,p,q}(t)
\]

\[
= \frac{1}{([n]_{p,q} + \beta)^2} \left[ p^{2n-2k} [k]_{p,q}^2 b^k_{n,p,q}(t) + 2\alpha p^{n-k} [k]_{p,q} b^k_{n,p,q}(t) + \alpha^2 b^k_{n,p,q}(t) \right]
\]

\[
= \frac{1}{([n]_{p,q} + \beta)^2} \left[ A + B + C \right].
\]

\[
A = p^{2n} \sum_{k=0}^{n} \frac{[k]^2}{p^{2k}} \prod_{j=1}^{n} \left\{ p^{j-1}(1-t) + q^{j-1}t \right\}
\]

\[
A = [n]p^{2n} \sum_{k=1}^{n} \frac{[k]}{p^{2k}} \prod_{j=1}^{n} \left\{ p^{j-1} + q^{j-1}u \right\}
\]

On shifting the limits and on replacing \( k \) by \( k + 1 \), we get

\[
A = [n]p^{2n} \sum_{k=1}^{n} \frac{[k+1]}{p^{2k+2}} \prod_{j=1}^{n-1} \left\{ p^j + q^j u \right\}
\]

\[
= [n]p^{n+u} \sum_{k=0}^{n-1} \frac{[k+1]}{p^{k+2}} \prod_{j=0}^{n-2} \left\{ p^j + q^j \left( \frac{2n}{p} \right) \right\}
\]
Using \([k + 1]_{p,q} = p^k + q[k]_{p,q}\), we get our desired result:

\[
A = [n]_{p}^{u} \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[p^k + q[k]]}{p^{k+2}} \frac{1}{p^{(n-k)(n-k-2)} q^{k(k+1)}} \left( \frac{qu}{p} \right)^k \prod_{j=0}^{n-2} \{p^j + q^j \left( \frac{qu}{p} \right) \},
\]

\[
eq [n]_{p,q}^{n-1} \frac{u}{u+1} + \frac{q^2 u^2 [n]_{p,q} [n-1]_{p,q}}{(u+1)(p+qu)},
\]
equivalently

\[
A = [n]_{p,q}^{n-1} x + \frac{q^2 [n]_{p,q} [n-1]_{p,q} x^2}{(p(1-x) + qx)}.
\]

Similarly

\[
B = 2 \alpha [n]_{p}^{n} \sum_{k=0}^{n} \frac{[k]}{p^k} \frac{1}{p^{(n-k)(n-k-1)} q^{k(k+1)}} \left( 1 - t \right)^{-k} \prod_{j=1}^{n} \{p^j - p^{-1} + q^{-1} t \},
\]

\[
eq 2 \alpha [n]_{p,q}^{n} \frac{1}{p^n} \sum_{k=1}^{n} \frac{1}{p^k} \left( \frac{qu}{p} \right)^k \prod_{j=1}^{n} \{p^j - p^{-1} + q^{-1} t \},
\]

After shifting the limits and on replacing \(k\) by \(k + 1\), we get

\[
B = 2 \alpha [n]_{p,q}^{n} \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p} \frac{1}{p^{(n-k)(n-k-2)} q^{k(k+1)}} \left( \frac{qu}{p} \right)^k \prod_{j=0}^{n-2} \{p^j + q^j \left( \frac{qu}{p} \right) \},
\]

which implies

\[
B = 2 \alpha [n]_{p,q}^{n} x.
\]

Similarly

\[
C = \alpha^2 \sum_{k=0}^{n} \frac{[1]}{p,q} \frac{1}{p^{(n-k)(n-k-1)} q^{k(k+1)}} \left( 1 - t \right)^{-k} \prod_{j=1}^{n} \{p^j - p^{-1} + q^{-1} t \},
\]

\[
eq \alpha^2.
\]
Theorem 2.1. Let \( 0 < q_n < p_n \leq 1 \) such that \( \lim_{n \to \infty} p_n = 1 \), \( \lim_{n \to \infty} q_n = 1 \), and for \( f \in C[0,1] \), we have \( \lim_n |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| = 0 \).

Proof. Let us recall the following Korovkin’s theorem see [20]. Let \((T_n)\) be a sequence of positive linear operators from \( C[a, b] \) into \( C[a, b] \). Then \( \lim_n \|T_n(f, x) - f(x)\|_{C[a,b]} = 0 \), for all \( f \in C[a, b] \) if and only if \( \lim_n \|T_n(f_i, x) - f_i(x)\|_{C[a,b]} = 0 \), for \( i = 0, 1, 2 \), where \( f_0(t) = 1 \), \( f_1(t) = t \) and \( f_2(t) = t^2 \).

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators \( L_{n,p,q}^{\alpha,\beta}(f;x) \) to the functions \( f \) by means of modulus of continuity, elements of Lipschitz class and peetre’s K-functional.

Let \( f \in C[0,1] \). The modulus of continuity of \( f \) denoted by \( \omega(f, \delta) \) is defined as:

\[
\omega(f, \delta) = \sup_{y, x \in [0,1], \, |y-x| < \delta} |f(y) - f(x)|.
\]

where \( w(f; \delta) \) satisfies the following conditions: for all \( f \in C[0,1] \),

\[
\lim_{\delta \to 0} w(f; \delta) = 0.
\]

and

\[
|f(y) - f(x)| \leq w(f; \delta) \left( \frac{|y-x|}{\delta} + 1 \right).
\]

Theorem 3.1. Let \( 0 < q < p \leq 1 \), and \( f \in C[0,1] \), and \( \delta > 0 \), we have

\[
\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n)
\]

where

\[
\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}} + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.
\]
Proof. From lemma (4.1) we have

\[ |L_{\alpha,\beta}^{n,p,q}(t - x)^2; x) = \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right)x^2 \]
\[ + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right)x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}. \]  

(3.3)

For \( x \in [0,1] \), we take

\[ |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left( L_{n,p,q}^{\alpha,\beta}(t - x)^2; x) \right)^{\frac{3}{2}} \right\}, \]

then we get

\[ \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left( \frac{1}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right)^{\frac{1}{2}} \right\}. \]

If we choose

\[ \delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}. \]

Then we have

\[ \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2w(f; \delta_n). \]

So we have the desired result. \( \square \)

Now we compute the approximation order of operator \( L_{n,p,q}^{\alpha,\beta} \) in term of the elements of the usual Lipschitz class.
Let $f \in C[0,1]$ and $0 < \rho \leq 1$. We recall that $f$ belongs to $\text{Lip}_M(\rho)$ if the inequality

$$|f(x) - f(y)| \leq M|x - y|\rho; \text{ for all } x, y \in [0,1]$$

holds.

Theorem 3.2. For all $f \in \text{Lip}_M(\rho)$

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq M\delta_n^\rho$$

where

$$\delta_n = \left[ \left( \frac{q^n[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \right]^{\frac{1}{2}}$$

and $M$ is a positive constant.

Proof. Let $f \in \text{Lip}_M(\rho)$ and $0 < \rho \leq 1$. by (3.4) and linearity and monotonicity of $L_{n,p,q}^{\alpha,\beta}$ then we have

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq L_{n,p,q}^{\alpha,\beta}(|f(t) - f(x)|; x)$$

$$\leq L_{n,p,q}^{\alpha,\beta}(|t - x|\rho; x).$$

Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq (L_{n,p,q}^{\alpha,\beta}((t - x)^2; x))^\frac{n}{m}.$$ (3.5)

if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L_{n,p,q}^{\alpha,\beta}$ by means of the Peetre’s K-functionals.

$C^2[0,1]$: The space of those functions $f$ for which $f, f', f'' \in C[0,1]$. we recall the following norm in the space $C^2[0,1]$

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$ 

We consider the following Peetre’s K-functional

$$K(f, \delta) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\|_{C[0,1]} + \delta\|g\|_{C^2[0,1]} \right\}.$$
Theorem 3.3. Let \( f \in C[0,1] \). Then we have
\[
\|L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f; \delta_n)
\]
Where \( K(f; \delta_n) \) is Peetre’s functional and
\[
\delta_n = \frac{1}{4} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q} - [n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} \right) + \frac{1}{4} \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{4([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{\alpha}{[n]_{p,q} + \beta}.
\]

Proof. Let \( g \in C^2[0,1] \). If we use the Taylor’s expansion of the function \( g \) at \( s = x \), we have
\[
g(s) = g(x) + (s-x)g'(x) + \frac{(s-x)^2}{2}g''(x).
\]
Hence we get
\[
\|L_{\alpha,\beta}^{n,p,q}(g; x) - g(x)\|_{C[0,1]} \leq \|L_{\alpha,\beta}^{n,p,q}((s-x); x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}
\]
\[
+ \frac{1}{2} \|L_{\alpha,\beta}^{n,p,q}((s-x)^2; x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}.
\]
(3.6)

From the lemma (2.1) we have
\[
\|L_{\alpha,\beta}^{n,p,q}((s-x); x)\|_{C[0,1]} \leq \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.
\]
(3.7)

So if we use (3.3) and (3.7) in (3.6), then we get
\[
\|L_{\alpha,\beta}^{n,p,q}(g; x) - g(x)\|_{C[0,1]} \leq \left[ \frac{1}{2} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q} - [n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} \right) + \frac{1}{2} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right] \|g(x)\|_{C[0,1]}.
\]
(3.8)

On the other hand, we can write
\[ |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq |L_{n,p,q}^{\alpha,\beta}(f - g; x)| + |L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)| + |f(x) - g(x)|. \]

If we take the maximum on \([0, 1]\), we have

\[ \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]}. \quad (3.10) \]

If we consider (3.8) in (3.10), we obtain

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \left[ \frac{1}{4} \frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} \right. \\
\left. - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} + \frac{1}{4} \frac{p^n-1[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right] + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]} \|g(x)\|_{C^2[0,1]}.
\]

If we choose

\[
\delta_n = \frac{1}{4} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \\
+ \left( \frac{1}{4} \frac{p^n-1[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]},
\]

then we get

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\left\{ \|f - g\|_{C[0,1]} + \delta_n \|g(x)\|_{C^2[0,1]} \right\}. \quad (3.11)
\]

Finally, one can observe that if we take the infimum of both side of above inequality for the function \(g \in C^2[0,1]\), we can find

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f, \delta_n).
\]

\[\square\]

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.
Approximation by $\left( p, q \right)$-Lupaş Stancu operators

Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence $x = (x_k)$ is said to be statistically convergent to a number $L$ if for every $\varepsilon > 0$,

$$
\delta\{K \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0,
$$

where $\delta(K)$ is the natural density of the set $K \subseteq \mathbb{N}$.

The density of subset $K \subseteq \mathbb{N}$ is defined by

$$
\delta(K) := \lim_{n \to \infty} \frac{1}{n} \{\text{the number } k \leq n : k \in K\}
$$

whenever the limit exists.

For instance, $\delta(\mathbb{N}) = 1$, $\delta\{2K : k \in \mathbb{N}\} = \frac{1}{2}$ and $\delta\{k^2 : K \in \mathbb{N}\} = 0$.

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$
X_k = \begin{cases} 
L_1; & \text{if } k = m^2, \\
L_2; & \text{if } k \neq m^2.
\end{cases}
\quad \text{where } m \in \mathbb{N}
$$

(4.1)

is statistically convergent to $L_2$ but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statistically convergent but not conversely.

Now we consider sequences $q = q_n$ and $p = p_n$ such that:

$$
st - \lim_{n \to \infty} q_n = 1, \quad st - \lim_{n \to \infty} p_n = 1, \quad \text{and} \quad st - \lim_{n \to \infty} q_n^a = 1.
$$

(4.2)

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.
Theorem 4.1. If $A_n$ be the sequence of linear positive operators from $C[a, b]$ to $C[a, b]$ satisfies the conditions
\[ \text{st} - \lim_{n} \| A_n((t^{\nu}; x)) - (x)\|_{C[0, 1]} = 0 \text{ for } \nu = 0, 1, 2. \]
then for any function $f \in C[a, b]$,
\[ \text{st} - \lim_{n} \| A_n(f; .) - f\|_{C[a, b]} = 0. \]

Now we will discuss the rates of statistical convergence of $L_{n,p,q}^{\alpha,\beta}$ operators.

Remark 4.2. For $q \in (0, 1)$ and $p \in (q, 1]$, it is obvious that
\[ \lim_{n \to \infty} [n]_{p,q} = \begin{cases} 0, & \text{when } p, q \in (0, 1) \\ \frac{1}{1-q}, & \text{when } p = 1 \text{ and } q \in (0, 1). \end{cases} \]

In order to reach to convergence results of the operator $L_{n,p,q}^{\alpha,\beta}(f; x)$, we take a sequence $q_n \in (0, 1)$ and $p_n \in (q_n, 1]$ such that $\lim_{n \to \infty} p_n = 1$, $\lim_{n \to \infty} q_n = 1$. So we get $\lim_{n \to \infty} [n]_{p_n,q_n} = \infty$.

Theorem 4.3. Let $L_{n,p,q}^{\alpha,\beta}$ be the sequence of operators and the sequences $p = p_n$ and $q = q_n$ satisfies Remark 4.2 then for any function $f \in C[0, 1]$
\[ \text{st} - \lim_{n} \| L_{n,p_n,q_n}^{\alpha,\beta}(f, .) - f\| = 0. \quad (4.3) \]

Proof. Clearly for $\nu = 0$,
\[ L_{n,p_n,q_n}^{\alpha,\beta}(1, x) = 1, \]
which implies
\[ \text{st} - \lim_{n} \| L_{n,p_n,q_n}^{\alpha,\beta}(1; x) - 1\| = 0. \]

For $\nu = 1$
For a given $\epsilon > 0$, let us define the following sets.

\[ U = \{ n : \| L^{\alpha,\beta}_{n,p,q_n}(t;x) - x \| \geq \epsilon \} \]
\[ U' = \{ n : 1 - \frac{[n]_{p,q_n}}{[n]_{p,q_n} + \beta} \geq \epsilon \} \]
\[ U'' = \{ n : \frac{\alpha}{[n]_{p,q_n} + \beta} \geq \epsilon \} . \]

It is obvious that $U \subseteq U'' \cup U'$.

So using

\[ \delta \{ k \leq n : 1 - \frac{[n]_{p,q_n}}{[n]_{p,q_n} + \beta} \geq \epsilon \} , \]

then we get

\[ \text{st-} \lim \frac{n}{\| L^{\alpha,\beta}_{n,p,q_n}(t;x) - x \|} = 0 . \tag{4.4} \]

Lastly for $\nu = 2$, we have

\[
\| L^{\alpha,\beta}_{n,p,q_n}(t^2 : x) - x^2 \| \leq \left| \frac{q^2[n]_{p,q_n}[n-1]_{p,q_n}}{p(1-x) + qx} \frac{1}{([n]_{p,q_n} + \beta)^2} - 1 \right| \\
+ \left| \frac{[n]_{p,q_n}(2\alpha + p^{n-1})}{([n]_{p,q_n} + \beta)^2} x \right| + \left| \frac{\alpha^2}{([n]_{p,q_n} + \beta)^2} \right| .
\]

If we choose

\[
\alpha_n = \frac{q^2[n]_{p,q_n}[n-1]_{p,q_n}}{p(1-x) + qx} \frac{1}{([n]_{p,q_n} + \beta)^2} - 1 \\
\beta_n = \frac{[n]_{p,q_n}(2\alpha + p^{n-1})}{[n]_{p,q_n} + \beta} \\
\gamma_n = \frac{\alpha^2}{([n]_{p,q_n} + \beta)^2} .
\]

Then

\[ \text{st-} \lim \frac{n}{\alpha_n} = \text{st-} \lim \frac{n}{\beta_n} = \text{st-} \lim \frac{n}{\gamma_n} = 0 . \]

Now given $\epsilon > 0$, we define the following four sets:

\[ U = \| L^{\alpha,\beta}_{n,p,q_n}(t^2 : x) - x^2 \| \geq \epsilon , \]
\[ U_1 = \{ n : \alpha_n \geq \frac{\epsilon}{3} \} , \]
\[ U_2 = \{ n : \beta_n \geq \frac{\epsilon}{3} \} , \]
\[ U_3 = \{ n : \gamma_n \geq \frac{\epsilon}{3} \} . \]
\( U_3 = \{ n : \gamma_n \geq \frac{\epsilon}{3} \}. \)

It is obvious that \( U \subseteq U_1 \cup U_2 \cup U_3. \) Thus we obtain

\[
\delta \{ K \leq n : \| L_{\alpha,\beta}^{\alpha,\beta}(t^2 : x) - x^2 \| \geq \epsilon \} \\
\leq \delta \{ K \leq n : \alpha_n \geq \frac{\epsilon}{3} \} + \delta \{ K \leq n : \beta_n \geq \frac{\epsilon}{3} \} + \delta \{ K \leq n : \gamma_n \geq \frac{\epsilon}{3} \}.
\]

So the right hand side of the inequalities is zero.

Then

\[
st - \lim_{n} \| L_{\alpha,\beta}^{\alpha,\beta}(t; x) - x \| = 0
\]

holds and thus the proof is completed.

\( \square \)

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