Approximation by \((p, q)\)-Lupaş Stancu Operators

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Abstract. In this paper, \((p, q)\)-Lupaş Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of \((p, q)\)-Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: \((p, q)\)-Integers, Lupaş \((p, q)\)-Bernstein Stancu operators, Statistical approximation, Korovkin’s type approximation.


1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators \(B_n : C[0, 1] \to C[0, 1]\) defined for any \(n \in \mathbb{N}\) and for any function \(f \in C[0, 1]\)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1].
\]

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].
Further, based on $q$-integers, Lupaş [21] introduced the first $q$-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another $q$-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on $q$-integers have been introduced and their approximation properties studied.

Recently, the applications of $(p, q)$-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving $(p, q)$-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen et al [27] introduced the concept of post quantum calculus in approximation theory and constructed the $(p, q)$-analogue of Bernstein operators defined as follows for $0 < q < p < 1$:

$$B_{n,p,q}(f; x) = \frac{1}{p^{(n-1)}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^k - q^k}\right), \ x \in [0, 1].$$

(1.2)

Note when $p = 1$, $(p, q)$-Bernstein Operators given by (1.2) turns out to be Phillips $q$-Bernstein Operators [38].

Also, we have

$$ (1 - x)^n_{p,q} = \prod_{k=0}^{n-1} (p^k - q^k x) = (1 - x)(p - qx)(p^2 - q^2 x)\ldots(p^{n-1} - q^{n-1} x)$$

$$= \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

Further, they applied the concept of $(p, q)$-calculus in approximation theory and studied approximation properties based on $(p, q)$-integers for Bernstein-Stancu operators, $(p, q)$-analogue of Bernstein-Kantorovich, $(p, q)$-analogue of Bernstein-Shurer operators, $(p, q)$-analogue of Bleimann-Butzer-Hahn operators and $(p, q)$-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined $(p, q)$-analogue of Lupaş Bernstein operators [17] as follows:
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For any \(p > 0\) and \(q > 0\), the linear operators \(L^{n}_{p,q} : C[0,1] \rightarrow C[0,1]\) as

\[
L^{n}_{p,q}(f;x) = \sum_{k=0}^{n} \binom{n}{k}_p q^{-k} \left( \frac{p^{n-k} [k]_{p,q}}{n!} \right) \left( \sum_{j=1}^{n} \prod_{j=1}^{n} \frac{p^{n-j} (1-x) + q^{1-j} x}{x} \right) x^k (1-x)^{n-k}.
\]

(1.3)

are \((p, q)\)-analogue of Lupaş Bernstein operators.

Again when \(p = 1\), Lupaş \((p, q)\)-Bernstein operators turns out to be Lupaş \(q\)-Bernstein operators as given in [22, 37].

When \(p = q = 1\), Lupaş \((p, q)\)-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau’s algorithm and Korovkin’s type approximation properties [17]: de-Casteljau’s algorithm and related results of degree elevation reduction for Bézier curves and surfaces holds for all \(p > 0\) and \(q > 0\). However to study Korovkin’s type approximation properties for Lupaş \((p, q)\)-analogue of the Bernstein operators, \(0 < q < p \leq 1\) is needed.

Based on Korovkin’s type approximation, they proved that the sequence of \((p, q)\)-analogue of Lupaş Bernstein operators \(L^{n}_{p,q} (f, x)\) converges uniformly to \(f(x) \in C[0,1]\) if and only if \(0 < q_n < p_n \leq 1\) such that \(\lim n \rightarrow \infty q_n = 1\), \(\lim n \rightarrow \infty p_n = 1\) and \(\lim n \rightarrow \infty q_n = 1\). On the other hand, for any \(p > 0\) fixed and \(p \neq 1\), the sequence \(L^{n}_{p,q} (f, x)\) converges uniformly to \(f(x) \in C[0,1]\) if and only if \(f(x) = ax + b\) for some \(a, b \in \mathbb{R}\).

Furthermore, in comparison to \(q\)-Bézier curves and surfaces based on Lupaş \(q\)-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter \(p\) have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of \((p, q)\)-calculus.

For any \(p > 0\) and \(q > 0\), the \((p, q)\) integers \([n]_{p,q}\) are defined by
\[ [n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, & \text{when } p \neq q \neq 1 \\ n \ p^{n-1}, & \text{when } p = q \neq 1 \\ [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases} \]

where \([n]_q\) denotes the \(q\)-integers and \(n = 0, 1, 2, \ldots\).

The formula for \((p, q)\)-binomial expansion is as follows:

\[
(ax + by)^n_{p,q} := \sum_{k=0}^{n} \binom{n}{k}_{p,q} \frac{\left(\frac{a}{q} - k\right)}{k} \left(\frac{b}{q} - k\right)^k x^n y^k.
\]

\[(x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y)\ldots(p^{n-1}x + q^{n-1}y),\]

\[(1 - x)^n_{p,q} = (1 - x)(p - qx)(p^2 - q^2x)\ldots(p^{n-1} - q^{n-1}x),\]

where \((p, q)\)-binomial coefficients are defined by

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}!\ [n-k]_{p,q}!}.
\]

Details on \((p, q)\)-calculus can be found in [10, 11, 27].

Also, we have \((p, q)\)-analogue of Euler’s identity as:

\[(1 - x)^n_{p,q} = \prod_{s=0}^{n-1} (p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2x)\ldots(p^{n-1} - q^{n-1}x)
\]

\[= \sum_{k=0}^{n} (-1)^k \frac{\left(\frac{a}{q} - k\right)}{k} \left(\frac{a}{q} - k\right)^k \binom{n}{k}_{p,q} x^k.\]

Again by some simple calculations and using the property of \((p, q)\)-integers, we get \((p, q)\)-analogue of Pascal’s relation as follows:

\[
\binom{n}{k}_{p,q} = q^{n-k} \binom{n-1}{k-1}_{p,q} + p^k \binom{n-1}{k}_{p,q}, \tag{1.4}
\]

\[
\binom{n}{k}_{p,q} = p^{n-k} \binom{n-1}{k-1}_{p,q} + q^k \binom{n-1}{k}_{p,q}. \tag{1.5}
\]

We recall some results from [17] for Lupas \((p, q)\)-Bernstein operators, which reproduces linear and constant functions.
Some auxiliary results:

(1) \( L_{n}^{p,q}(1, \frac{u}{u+1}) = 1 \)

(2) \( L_{n}^{p,q}(t, \frac{u}{u+1}) = \frac{u}{u+1} \)

(3) \( L_{n}^{p,q}(t^2, \frac{u}{u+1}) = \frac{u}{u+1} \cdot p^{n-1} + \frac{qu}{u+1} \cdot \left( \frac{n-1}{[n]_{p,q}} \right) \)

or equivalently for \( x = \frac{u}{u+1} \)

\[
L_{n}^{p,q}(1, x) = 1, \\
L_{n}^{p,q}(t, x) = x, \\
L_{n}^{p,q}(t^2; x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^n x^2}{p(1-x) + qx} \cdot \left( \frac{n-1}{[n]_{p,q}} \right).
\]  

2. Construction of \((p, q)\)-Lupaş Stancu Operators

In this section, we introduce \((p, q)\)-Lupaş Stancu operators as follows:

For any \( p > 0 \) and \( q > 0 \), the linear operators \( L_{n}^{p,q} : C[0,1] \rightarrow C[0,1] \)

\[
L_{n}^{\alpha,\beta}_{n,p,q}(f; x) = \sum_{k=0}^{n} f \left( \frac{p^{n-k}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{k,n}^{p,q}(t)
\]

and \( b_{k,n}^{p,q}(t) \) is given by

\[
b_{k,n}^{p,q}(t) = \binom{n}{k} \frac{p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{ p^{j-1}(1-t) + q^{j-1}t \}^{2}}.
\]

where \( 0 < \alpha < \beta \).

We give some equalities for operators (2.1) in the following lemma.

**Lemma 4.1.** The following equalities are true:

(i) \( L_{n}^{\alpha,\beta}_{n,p,q}(1; x) = 1 \),

(ii) \( L_{n}^{\alpha,\beta}_{n,p,q}(t; x) = \frac{[n]_{p,q}t^{\alpha}}{[n]_{p,q} + \beta} \),

(iii) \( L_{n}^{\alpha,\beta}_{n,p,q}(t^2; x) = \frac{1}{[n]_{p,q} + \beta} \cdot \frac{q^n x^2}{p(1-x) + qx} \cdot x^2 + \frac{[n]_{p,q}(2^n + p^n - 1)}{[n]_{p,q} + \beta} x + \frac{\alpha^2}{[n]_{p,q} + \beta} x. \)
Proof. Proof of part (i) is obvious.

\[ L_{n,p,q}^{\alpha,\beta}(t; x) = \sum_{k=0}^{n} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b^{k,n}_{p,q}(t) \]

\[ = \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(t; x) + \frac{[\alpha]}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(1; x). \]

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

\[ L_{n,p,q}^{\alpha,\beta}(t^2; x) = \sum_{k=0}^{n} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b^{k,n}_{p,q}(t) \]

\[ = \frac{1}{([n]_{p,q} + \beta)^2} \left[ p^{2n-2k} [k]_{p,q}^2 b^{k,n}_{p,q}(t) \right. \]

\[ + \left. 2\alpha p^{n-k} [k]_{p,q} b^{k,n}_{p,q}(t) + \alpha^2 b^{k,n}_{p,q}(t) \right] \]

\[ = \frac{1}{([n]_{p,q} + \beta)^2} \left[ A + B + C \right]. \]

\[ A = p^{2n} \sum_{k=0}^{n} \binom{n}{k}^2 \frac{p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}} \]

\[ A = [n]p^{2n} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k+1)}{2}} u^k}{\prod_{j=1}^{n} \{p^j + q^j u\}}. \]

On shifting the limits and on replacing \( k \) by \( k + 1 \), we get

\[ A = [n]p^{2n} \sum_{k=1}^{n} \binom{n-1}{k+1} \frac{p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k+1)}{2}} u^k}{\prod_{j=1}^{n-1} \{p^j + q^j u\}} \]

\[ = [n]p^{n} \sum_{k=0}^{n-1} \binom{n-1}{k+1} \frac{p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k+1)}{2}} (\frac{u}{p})^k}{\prod_{j=0}^{n-2} \{p^j + q^j (\frac{u}{p})\}}. \]
Using \([k+1]_{p,q} = p^k + q[k]_{p,q}\), we get our desired result:

\[
A = [n]_{p,q}^u \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[p^k + q[k]]}{p^{k+2}} \sum_{j=0}^{n-2} \left( p^j + q^j \left( \frac{q[u]}{p} \right) \right) 
\]

\[
= [n]_{p,q}^n \frac{u}{u+1} + \frac{q^2 u^2 [n]_{p,q} [n-1]_{p,q}}{(u+1)(p+q\alpha)} 
\]
equivalently

\[
A = [n]_{p,q}^n \frac{u}{u+1} + \frac{q^2 u^2 [n]_{p,q} [n-1]_{p,q}}{(p+q\alpha)^2} 
\]

Similarly

\[
B = 2\alpha [n]_{p,q}^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[k]}{p^k} \sum_{j=1}^{n-1} \frac{p^j}{(1-t)^n} \prod_{j=1}^{n-1} \left( p^j + q^j \right) 
\]

\[
= 2\alpha [n]_{p,q}^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p^k} \sum_{j=1}^{n-1} \frac{p^j}{(1-t)^n} \prod_{j=1}^{n-1} \left( p^j + q^j \right) 
\]

After shifting the limits and on replacing \(k\) by \(k+1\), we get

\[
B = 2\alpha [n]_{p,q}^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p^k} \sum_{j=1}^{n-1} \frac{p^j}{(1-t)^n} \prod_{j=1}^{n-1} \left( p^j + q^j \right) 
\]

which implies

\[
B = 2\alpha [n]_{p,q} x 
\]

Similarly

\[
C = \alpha^2 \sum_{k=0}^{n} \frac{[k]}{p^k} \sum_{j=1}^{n} \frac{p^j}{(1-t)^n} \prod_{j=1}^{n} \left( p^j + q^j \right) 
\]

\[
= \alpha^2 
\]
Theorem 2.1. Let $0 < q_n < p_n \leq 1$ such that $\lim_{n \to \infty} p_n = 1$, $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} q_n^n = 1$ and for $f \in C[0,1]$, we have $\lim_n |L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)| = 0$.

Proof. Let us recall the following Korovkin’s theorem see [20]. Let $(T_n)$ be a sequence of positive linear operators from $C[0,1]$ into $C[0,1]$. Then $\lim_{n} \|T_n(f, x) - f(x)\|_{C[0,1]} = 0$, for all $f \in C[0,1]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_{C[0,1]} = 0$, for $i = 0, 1, 2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators $L_{\alpha,\beta}^{n,p,q}(f; x)$ to the functions $f$ by means of modulus of continuity, elements of Lipschitz class and peetre’s K-functional.

Let $f \in C[0,1]$. The modulus of continuity of $f$ denoted by $\omega(f, \delta)$ is defined as:

$$\omega(f, \delta) = \sup_{y, x \in [0,1], |y - x| < \delta} |f(y) - f(x)|.$$ 

where $w(f; \delta)$ satisfies the following conditions: for all $f \in C[0,1]$,

$$\lim_{\delta \to 0} w(f; \delta) = 0. \quad (3.1)$$

and

$$|f(y) - f(x)| \leq w(f; \delta) \left( \frac{|y - x|}{\delta} + 1 \right). \quad (3.2)$$

Theorem 3.1. Let $0 < q < p \leq 1$, and $f \in C[0,1]$, and $\delta > 0$, we have

$$\|L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n)$$

where

$$\delta_n = \left[ \left( \frac{q^n[n]_p}[n-1]_p - \frac{[n]_p - \beta}{[n]_p + \beta} \right) + \frac{p^n[n]_p - 2\alpha\beta}{([n]_p + \beta)^2} \right]^{\frac{1}{2}}.$$
Proof. From lemma (4.1) we have

\[
|L_{n,p,q}^{\alpha,\beta}(t-x)^2; x) = \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2} x^2 \\
+ \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.
\] (3.3)

For \( x \in [0,1] \), we take

\[
|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq w(f; \delta) \left( 1 + \frac{1}{\delta} (L_{n,p,q}^{\alpha,\beta}(t-x)^2; x) \right)^{\frac{1}{2}},
\]

then we get

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq w(f; \delta) \left( 1 + \frac{1}{\delta} \left( \frac{1}{([n]_{p,q} + \beta)^2} - \frac{[n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2} + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) \right)^{\frac{1}{2}} \right).
\]

If we choose

\[
\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right)^{\frac{1}{2}} \right].
\]

Then we have

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n).
\]

So we have the desired result. \( \square \)

Now we compute the approximation order of operator \( L_{n,p,q}^{\alpha,\beta} \) in term of the elements of the usual Lipschitz class.
Let $f \in C[0,1]$ and $0 < \rho \leq 1$. We recall that $f$ belongs to $\text{Lip}_M(\rho)$ if the inequality

$$|f(x) - f(y)| \leq M|x - y|^{\rho}; \text{ for all } x, y \in [0,1] \quad (3.4)$$

holds.

**Theorem 3.2.** For all $f \in \text{Lip}_M(\rho)$

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq M\delta_n$$

where

$$\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right)^{\frac{1}{2}} \right]$$

and $M$ is a positive constant.

**Proof.** Let $f \in \text{Lip}_M(\rho)$ and $0 < \rho \leq 1$, by (3.4) and linearity and monotonicity of $L_{n,p,q}^{\alpha,\beta}$ then we have

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq L_{n,p,q}^{\alpha,\beta}(|f(t) - f(x)|; x)$$

$$\leq L_{n,p,q}^{\alpha,\beta}(|t-x|^{\rho}; x).$$

Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq (L_{n,p,q}^{\alpha,\beta}((t-x)^{\rho}; x))^{\frac{2}{\rho}}. \quad (3.5)$$

if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L_{n,p,q}^{\alpha,\beta}$ by means of the Peetre’s K-functionals.

$C^2[0,1]$: The space of those functions $f$ for which $f, f', f'' \in C[0,1]$, we recall the following norm in the space $C^2[0,1]$

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$  

We consider the following Peetre’s K-functional

$$K(f, \delta) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\|_{C[0,1]} + \delta\|g\|_{C^2[0,1]} \right\}.$$
Theorem 3.3. Let \( f \in C[0,1] \). Then we have
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f; \delta_n)
\]
where \( K(f; \delta_n) \) is Peetre’s functional and
\[
\delta_n = \frac{1}{4} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \frac{1}{4} \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \left( \frac{[n]_{p,q} + \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.
\]

Proof. Let \( g \in C^2[0,1] \). If we use the Taylor’s expansion of the function \( g \) at \( s = x \), we have
\[
g(s) = g(x) + (s - x)g'(x) + \frac{(s - x)^2}{2}g''(x).
\]
Hence we get
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq \|L_{n,p,q}^{\alpha,\beta}((s - x); x)\|_{C[0,1]}\|g(x)\|_{C[0,1]}
\]
\[
+ \left( \frac{1}{2} \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]} \|g(x)\|_{C[0,1]}.
\]

(3.6)

From the lemma (2.1) we have
\[
\|L_{n,p,q}^{\alpha,\beta}((s - x); x)\|_{C[0,1]} \leq \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.
\]

(3.7)

So if we use (3.3) and (3.7) in (3.6), then we get
\[
\|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]} \leq \frac{1}{2} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{1}{2} \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]} \|g(x)\|_{C[0,1]}.
\]

(3.8)

On the other hand, we can write
\[ |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq |L_{n,p,q}^{\alpha,\beta}(f - g; x)| + |L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)| + |f(x) - g(x)|. \]

If we take the maximum on \([0, 1]\), we have
\[ \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0, 1]} \leq 2\|f - g\|_{C[0, 1]} + \|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0, 1]}. \] \tag{3.10}

If we consider (3.8) in (3.10), we obtain
\[
\begin{align*}
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0, 1]} & \leq 2\|f - g\|_{C[0, 1]} + \left[ \frac{1}{4} \left( q[n]_{p,q}[n - 1]_{p,q} \right) - \frac{n}{p} \beta \right] + \left[ \frac{1}{4} \left( \frac{1}{p} p^n - \beta \right) \right] + \frac{1}{2} \alpha \left[ \frac{1}{2} \alpha + \beta \right] \|g(x)\|_{C^2[0, 1]}.
\end{align*}
\]

If we choose
\[
\delta_n = \left[ \frac{1}{4} \left( q[n]_{p,q}[n - 1]_{p,q} \right) - \frac{n}{p} \beta \right] + \left[ \frac{1}{4} \left( \frac{1}{p} p^n - \beta \right) \right] + \frac{1}{2} \alpha \left[ \frac{1}{2} \alpha + \beta \right],
\]
then we get
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0, 1]} \leq 2\left\{ \|f - g\|_{C[0, 1]} + \delta_n \|g(x)\|_{C^2[0, 1]} \right\}. \tag{3.11}
\]

Finally, one can observe that if we take the infimum of both sides of above inequality for the function \(g \in C^2[0, 1]\), we can find
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0, 1]} \leq 2K(f, \delta_n).
\]

\[ \square \]

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadgjev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.
Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence $x = (x_k)$ is said to be statistically convergent to a number $L$ if for every $\epsilon > 0$,

$$\delta\{K \subseteq \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0,$$

where $\delta(K)$ is the natural density of the set $K \subseteq \mathbb{N}$.

The density of subset $K \subseteq \mathbb{N}$ is defined by

$$\delta(K) := \lim_{n \to \infty} \frac{1}{n} \{\text{the number } k \leq n : k \in K\}$$

whenever the limit exists.

For instance, $\delta(\mathbb{N}) = 1$, $\delta\{2k : k \in \mathbb{N}\} = \frac{1}{2}$ and $\delta\{k^2 : k \in \mathbb{N}\} = 0$.

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$X_k = \begin{cases} L_1; & \text{if } k = m^2, \\ L_2; & \text{if } k \neq m^2. \end{cases} \quad \text{where } m \in \mathbb{N} \tag{4.1}$$

is statistically convergent to $L_2$ but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statistically convergent but not conversely.

Now we consider sequences $q = q_n$ and $p = p_n$ such that:

$$st - \lim_n q_n = 1, \quad st - \lim_n p_n = 1, \quad \text{and} \quad st - \lim_n q_n^p = 1. \tag{4.2}$$

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.
Theorem 4.1. If \( A_n \) be the sequence of linear positive operators from \( C[a, b] \) to \( C[a, b] \) satisfies the conditions
\[
\text{st} - \lim_n \| A_n((t^\nu; x)) - (x)^\nu \|_{C[0, 1]} = 0 \quad \text{for} \nu = 0, 1, 2.
\]
then for any function \( f \in C[a, b] \),
\[
\text{st} - \lim_n \| A_n(f; .) - f \|_{C[a, b]} = 0.
\]

Now we will discuss the rates of statistical convergence of \( L_{\alpha,\beta}^{n,p,q} \) operators.

Remark 4.2. For \( q \in (0, 1) \) and \( p \in (q, 1] \), it is obvious that
\[
\lim_{n \to \infty} [n]_{p,q} = \begin{cases} 0, \text{ when } p, q \in (0, 1) \\ \frac{1}{1-q}, \text{ when } p = 1 \text{ and } q \in (0, 1). \end{cases}
\]

In order to reach to convergence results of the operator \( L_{\alpha,\beta}^{n,p,q}(f; x) \), we take a sequence \( q_n \in (0, 1) \) and \( p_n \in (q_n, 1] \) such that \( \lim_{n \to \infty} p_n = 1 \), \( \lim_{n \to \infty} q_n = 1 \). So we get \( \lim_{n \to \infty} [n]_{p_n,q_n} = \infty \).

Theorem 4.3. Let \( L_{\alpha,\beta}^{\alpha,\beta} \) be the sequence of operators and the sequences \( p = p_n \) and \( q = q_n \) satisfies Remark 4.2 then for any function \( f \in C[0, 1] \)
\[
\text{st} - \lim_n \| L_{\alpha,\beta}^{\alpha,\beta}(f; .) - f \| = 0. \quad (4.3)
\]

Proof. Clearly for \( \nu = 0 \),
\[
L_{\alpha,\beta}^{\alpha,\beta}(1, x) = 1,
\]
which implies
\[
\text{st} - \lim_n \| L_{\alpha,\beta}^{\alpha,\beta}(1; x) - 1 \| = 0.
\]

For \( \nu = 1 \)
\[
\| L_{\alpha,\beta}^{\alpha,\beta}(t; x) - x \| \leq \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} x - x \right| + \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right| \leq \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right| + \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right|.
\]
For a given $\epsilon > 0$, let us define the following sets.

\[
U = \{ n : \| L_{n,p_n,q_n}^\alpha,\beta(t;x) - x \| \geq \epsilon \}
\]

\[
U' = \{ n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \geq \epsilon \}
\]

\[
U'' = \{ n : \frac{\alpha}{[n]_{p_n,q_n} + \beta} \geq \epsilon \}
\]

It is obvious that $U \subseteq U'' \cup U'$.

So using

\[
\delta \{ k \leq n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \geq \epsilon \},
\]

then we get

\[
st - \lim_n \| L_{n,p_n,q_n}^\alpha,\beta(t;x) - x \| = 0.
\]  \hspace{1cm} (4.4)

Lastly for $\nu = 2$, we have

\[
\| L_{n,p_n,q_n}^\alpha,\beta(t^2 : x) - x^2 \| \leq \left| \frac{q^2[n]_{p_n,q_n}([n-1]_{p_n,q_n}}{p(1-x) + qx} \left( \frac{1}{([n]_{p_n,q_n} + \beta)^2} - 1 \right)
\right|
\]

\[
\left| + \frac{[n]_{p_n,q_n}(2\alpha + p^{n-1})^2}{x} + \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2} \right|.
\]

If we choose

\[
\alpha_n = \frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x) + qx} \left( \frac{1}{([n]_{p_n,q_n} + \beta)^2} - 1 \right)
\]

\[
\beta_n = \frac{[n]_{p_n,q_n}(2\alpha + p^{n-1})^2}{[n]_{p_n,q_n} + \beta}
\]

\[
\gamma_n = \frac{\alpha^2}{([n]_{p_n,q_n} + \beta)^2}.
\]

Then

\[
st - \lim_n \alpha_n = st - \lim_n \beta_n = st - \lim_n \gamma_n = 0.
\]

Now given $\epsilon > 0$, we define the following four sets:

\[
U = \| L_{n,p_n,q_n}^\alpha,\beta(t^2 : x) - x^2 \| \geq \epsilon,
\]

\[
U_1 = \{ n : \alpha_n \geq \frac{\epsilon}{3} \},
\]

\[
U_2 = \{ n : \beta_n \geq \frac{\epsilon}{3} \},
\]
\[ U_3 = \{ n : \gamma_n \geq \frac{\epsilon}{3} \}. \]

It is obvious that \( U \subseteq U_1 \bigcup U_2 \bigcup U_3 \). Thus we obtain

\[
\delta\{ K \leq n : \| L_{n,p,q}^{\alpha,\beta}(t^2 : x) - x^2 \| \geq \epsilon \} \\
\leq \delta\{ K \leq n : \alpha_n \geq \frac{\epsilon}{3} \} + \delta\{ K \leq n : \beta_n \geq \frac{\epsilon}{3} \} + \delta\{ K \leq n : \gamma_n \geq \frac{\epsilon}{3} \}.
\]

So the right hand side of the inequalities is zero.

Then

\[
\text{st - lim}_{n} \| L_{n,p,n,q}^{\alpha,\beta}(t ; x) - x \| = 0
\]

holds and thus the proof is completed.

\[ \square \]

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**References**


