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Approximation by (p, q)-Lupaş Stancu Operators

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Abstract. In this paper, (p, q)-Lupas Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of (p, q)-Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: (p, q)-Integers, Lupaş (p, q)-Bernstein Stancu operators, Statistical approximation, Korovkin's type approximation.

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1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators $B_n : C[0, 1] \to C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right), \; x \in [0, 1]. \tag{1.1}$$

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].

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Further, based on q-integers, Lupaş [21] introduced the first q-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another q-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on q-integers have been introduced and their approximation properties studied.

Recently, the applications of (p, q)-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving (p, q)-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen *et al* [27] introduced the concept of post quantum calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators defined as follows for $0 < q < p \leq 1$:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x)\, f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \right),\ x \in [0,1]. \tag{1.2}$$

Note when $p = 1$, (p, q)-Bernstein Operators given by (1.2) turns out to be Phillips q-Bernstein Operators [38].

Also, we have

$$\begin{aligned} (1-x)^n_{p,q} &= \prod_{s=0}^{n-1} (p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x) \\ &= \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k. \end{aligned}$$

Further, they applied the concept of (p, q)-calculus in approximation theory and studied approximation properties based on (p, q)-integers for Bernstein-Stancu operators, (p, q)-analogue of Bernstein-Kantorovich, (p, q)-analogue of Bernstein-Shurer operators, (p, q)-analogue of Bleimann-Butzer-Hahn operators and (p, q)-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined (p, q)-analogue of Lupaş Bernstein operators [17] as follows :

For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0, 1] \to C[0, 1]$ as

$$L^n_{p,q}(f; x) = \sum_{k=0}^{n} \frac{f\left(\frac{p^{n-k}\ [k]_{p,q}}{[n]_{p,q}}\right) \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\ x^k\ (1-x)^{n-k}}{\prod\limits_{j=1}^{n} \{p^{j-1}(1-x) + q^{j-1}x\}}, \tag{1.3}$$

are (p, q)-analogue of Lupaş Bernstein operators.

Again when $p = 1$, Lupaş (p, q)-Bernstein operators turns out to be Lupaş q-Bernstein operators as given in [22, 37].

When $p = q = 1$, Lupaş (p, q)-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau's algorithm and Korovkin's type approximation properties [17]: de-Casteljau's algorithm and related results of degree elevation reduction for Bèzier curves and surfaces holds for all $p > 0$ and $q > 0$. However to study Korovkin's type approximation properties for Lupaş (p, q)-analogue of the Bernstein operators, $0 < q < p \leq 1$ is needed.

Based on Korovkin's type approximation, they proved that the sequence of (p, q)-analogue of Lupaş Bernstein operators $L^n_{p_n,q_n}(f, x)$ converges uniformly to $f(x) \in C[0, 1]$ if and only if $0 < q_n < p_n \leq 1$ such that $\lim\limits_{n\to\infty} q_n = 1$, $\lim\limits_{n\to\infty} p_n = 1$ and $\lim\limits_{n\to\infty} q^n_n = 1$. On the other hand, for any $p > 0$ fixed and $p \neq 1$, the sequence $L^n_{p,q}(f, x)$ converges uniformly to $f(x) \in C[0, 1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Furthermore, in comparison to q-Bèzier curves and surfaces based on Lupaş q-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter p have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of (p, q)-calculus.

For any $p > 0$ and $q > 0$, the (p, q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1}+p^{n-2}q+p^{n-3}q^2+...+pq^{n-2}+q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q}, \text{ when } p \neq q \neq 1 \\ n\ p^{n-1}, \text{ when } p = q \neq 1 \\ [n]_q, \text{ when } p = 1 \\ n, \text{ when } p = q = 1 \end{cases}$$

where $[n]_q$ denotes the q-integers and $n = 0, 1, 2, \cdots$.

The formula for (p, q)-binomial expansion is as follows:

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k}b^k x^{n-k}y^k,$$

$$(x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$

$$(1 - x)^n_{p,q} = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x),$$

where (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n - k]_{p,q}!}.$$

Details on (p, q)-calculus can be found in [10, 11, 27].

Also, we have (p, q)-analogue of Euler's identity as:

$$(1 - x)^n_{p,q} = \prod_{s=0}^{n-1}(p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2x)...(p^{n-1} - q^{n-1}x)$$

$$= \sum_{k=0}^{n}(-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

Again by some simple calculations and using the property of (p, q)-integers, we get (p, q)-analogue of Pascal's relation as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \tag{1.4}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}. \tag{1.5}$$

We recall some results from [17] for Lupas (p, q)-Bernstein operators, which reproduces linear and constant functions.

Some auxillary results:

(1) $L^n_{p,q}(1, \frac{u}{u+1}) = 1$

(2) $L^n_{p,q}(t, \frac{u}{u+1}) = \frac{u}{u+1}$

(3) $L^n_{p,q}(t^2, \frac{u}{u+1}) = \frac{u}{u+1}\frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1}(\frac{qu}{p+qu})\frac{[n-1]_{p,q}}{[n]_{p,q}}$

or equivalently for $x = \frac{u}{u+1}$

$$L^n_{p,q}(1,x) = 1, \tag{1.6}$$

$$L^n_{p,q}(t,x) = x, \tag{1.7}$$

$$L^n_{p,q}(t^2,x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2\ x^2}{p(1-x)+qx}\frac{[n-1]_{p,q}}{[n]_{p,q}}. \tag{1.8}$$

2. Construction of (p,q)-Lupaş Stancu Operators

In this section, we introduce (p,q)-Lupaş Stancu operators as follows:

For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0,1] \to C[0,1]$

$$L^{\alpha,\beta}_{n,p,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{p^{n-k}\ [k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right) b^{k,n}_{p,q}(t) \tag{2.1}$$

and $b^{k,n}_{p,q}(t)$ is given by

$$b^{k,n}_{p,q}(t) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\ t^k\ (1-t)^{n-k}}{\prod_{j=1}^{n}\{p^{j-1}(1-t)+q^{j-1}t\}}, \tag{2.2}$$

where $0 < \alpha < \beta$.

We give some equalities for operators (2.1) in the following lemma.

Lemma 4.1. The following equalities are true:

(i) $L^{\alpha,\beta}_{n,p,q}(1;x) = 1$,

(ii) $L^{\alpha,\beta}_{n,p,q}(t;x) = \frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}$,

(iii) $L^{\alpha,\beta}_{n,p,q}(t^2;x) = \frac{1}{([n]_{p,q}+\beta)^2}\frac{q^2[n]_{p,q}[n-1]_{p,q}}{p(1-x)+qx}x^2 + \frac{[n]_{p,q}(2\alpha+p^{n-1})}{([n]_{p,q}+\beta)^2}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}$.

Proof. Proof of part (i) is obvious.

$$L^{\alpha,\beta}_{n,p,q}(t;x) = \sum_{k=0}^{n}\left(\frac{p^{n-k}\ [k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right) b^{k,n}_{p,q}(t)$$

$$= \frac{[n]_{p,q}}{[n]_{p,q}+[\beta]} L^{n}_{p,q}(t;x) \ + \ \frac{[\alpha]}{[n]_{p,q}+[\beta]} L^{n}_{p,q}(1;x).$$

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

$$L^{\alpha,\beta}_{n,p,q}(t^2;x) = \sum_{k=0}^{n}\left(\frac{p^{n-k}\ [k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right) b^{k,n}_{p,q}(t)$$

$$= \frac{1}{([n]_{p,q} \ + \ \beta)^2}\Big[p^{2n-2k}[k]^2_{p,q} b^{k,n}_{p,q}(t)$$

$$+ \ 2\alpha p^{n-k}[k]_{p,q} b^{k,n}_{p,q}(t) \ + \ \alpha^2 b^{k,n}_{p,q}(t)\Big]$$

$$= \frac{1}{([n]_{p,q} \ + \ \beta)^2}\Big[\ A \ + \ B \ + \ C\ \Big].$$

$$A \ = \ p^{2n}\sum_{k=0}^{n}\frac{[k]^2}{p^{2k}}\frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\ t^k\ (1-t)^{n-k}}{\prod_{j=1}^{n}\{p^{j-1}(1-t)+q^{j-1}t\}}$$

$$A \ = \ [n]p^{2n}\sum_{k=1}^{n}\frac{[k]}{p^{2k}}\frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\ u^k}{\prod_{j=1}^{n}\{p^{j-1}+q^{j-1}u\}}.$$

On shifting the limits and on replacing k by $k+1$, we get

$$A \ = \ [n]p^{2n}\sum_{k=1}^{n}\frac{[k+1]}{p^{2k+2}}\frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}}\ u^k}{\prod_{j=1}^{n-1}\{p^{j}+q^{j}u\}},$$

$$= \ [n]p^{n}\frac{u}{u+1}\sum_{k=0}^{n-1}\frac{[k+1]}{p^{k+2}}\frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}}\ (\frac{qu}{p})^k}{\prod_{j=0}^{n-2}\{p^{j}+q^{j}(\frac{qu}{p})\}}.$$

Using $[k+1]_{p,q} = p^k + q[k]_{p,q}$, we get our desired result:

$$\begin{aligned} A &= [n]p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[p^k + q[k]]}{p^{k+2}} \frac{\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k+1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j \left(\frac{qu}{p}\right)\}}, \\ &= [n]_{p,q} p^{n-1} \frac{u}{u+1} + \frac{q^2 u^2 [n]_{p,q} [n-1]_{p,q}}{(u+1)(p+qu)}, \end{aligned}$$

equivalently

$$A = [n]_{p,q} p^{n-1} x + \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{(p(1-x)+qx)} x^2.$$

Similarly

$$\begin{aligned} B &= 2\alpha p^n \sum_{k=0}^{n} \frac{[k]}{p^k} \frac{\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1} t\}}, \\ &= 2\alpha [n] p^n \sum_{k=1}^{n} \frac{1}{p^k} \frac{\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} u^k}{\prod_{j=1}^{n} \{p^{j-1} + q^{j-1} u\}}. \end{aligned}$$

After shifting the limits and on replacing k by $k+1$, we get

$$B = 2\alpha [n] p^n \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{1}{p} \frac{\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^{\frac{(n-k-1)(n-k-2)}{2}} q^{\frac{k(k-1)}{2}} \left(\frac{qu}{p}\right)^k}{\prod_{j=0}^{n-2} \{p^j + q^j \left(\frac{qu}{p}\right)\}},$$

which implies

$$B = 2\alpha [n]_{p,q} x.$$

Similarly

$$\begin{aligned} C &= \alpha^2 \sum_{k=0}^{n} \frac{\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1} t\}}, \\ &= \alpha^2. \end{aligned}$$

□

Theorem 2.1. *Let* $0 < q_n < p_n \leq 1$ *such that* $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = 1$ *and for* $f \in C[0,1]$, *we have* $\lim_n |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| = 0$.

Proof. Let us recall the following Korovkin's theorem see [20]. Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{\mathcal{C}[a,b]} = 0$, for $i = 0,1,2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

□

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators $L_{n,p,q}^{\alpha,\beta}(f;x)$ to the functions f by means of modulus of continuity, elements of Lipschitz class and peetre's K-functional.

Let $f \in C[0,1]$. The modulus of continuity of f denoted by $\omega(f,\delta)$ is defined as:

$$\omega(f,\delta) = \sup_{y,x\in[0,1],\ |y-x|<\delta} |f(y) - f(x)|.$$

where $w(f;\delta)$ satisfies the following conditions: for all $f \in C[0,1]$,

$$\lim_{\delta\to 0}\ w(f;\delta) = 0. \tag{3.1}$$

and

$$|f(y) - f(x)| \leq w(f;\delta)\bigg(\frac{|y-x|}{\delta} + 1\bigg). \tag{3.2}$$

Theorem 3.1. *Let* $0 < q < p \leq 1$, *and* $f \in C[0,1]$, *and* $\delta > 0$, *we have*

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \leq 2\omega(f;\delta_n)$$

where

$$\delta_n = \Bigg[\bigg(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\ +\ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q}\ +\ \beta}\bigg) + \bigg(\frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q}+\beta)^2}\bigg) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\Bigg]^{\frac{1}{2}}.$$

Proof. From lemma (4.1) we have

$$|L^{\alpha,\beta}_{n,p,q}(t-x)^2;x) = \left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\right)x^2 + \left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right)x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}. \tag{3.3}$$

For $x \in [0,1]$, we take

$$|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)| \leq w(f;\delta)\left\{1 + \frac{1}{\delta}(L^{\alpha,\beta}_{n,p,q}(t-x)^2 : x)^{\frac{1}{2}}\right\},$$

then we get

$$\begin{aligned}\|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)\|_{C[0,1]} &\leq w(f;\delta)\left\{1 + \frac{1}{\delta}(L^{\alpha,\beta}_{n,p,q}(t-x)^2 : x)^{\frac{1}{2}}\right\} \\ &\leq w(f;\delta)\left\{1 + \frac{1}{\delta}\left((\frac{1}{([n]_{p,q}+\beta)^2}\frac{q^2[n]_{p,q}[n-1]_{p,q}}{(p(1-x)+qx)}\right.\right. \\ &\quad - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}) \ + \ (\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}) \\ &\quad \left.\left. + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

If we choose

$$\delta_n = \left[\left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\right) + \left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\right]^{\frac{1}{2}}.$$

Then we have

$$\|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)\|_{C[0,1]} \ \leq \ 2\omega(f;\delta_n).$$

So we have the desired result. □

Now we compute the approximation order of operator $L^{\alpha,\beta}_{n,p,q}$ in term of the elements of the usual Lipschitz class.

Let $f \in C[0,1]$ and $0 < \rho \leq 1$. We recall that f belongs to $Lip_M(\rho)$ if the inequality

$$|f(x) - f(y)| \leq M|x-y|^{\rho};\ for\ all\ x, y \in [0,1] \tag{3.4}$$

holds.

Theorem 3.2. *For all $f \in Lip_M(\rho)$*

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \ \leq \ M\delta_n^{\rho}$$

where

$$\delta_n = \Bigg[\Bigg(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\Bigg) + \left(\frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q}+\beta)^2}\right) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\Bigg]^{\frac{1}{2}}$$

and M is a positive constant.

Proof. Let$f \in Lip_M(\rho)$ and $0 < \rho \leq 1$. by (3.4) and linearity and monotonicity of $L_{n,p,q}^{\alpha,\beta}$ then we have

$$\begin{aligned} |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| &\leq L_{n,p,q}^{\alpha,\beta}(|f(t)-f(x)|;x) \\ &\leq L_{n,p,q}^{\alpha,\beta}(|t-x|^{\rho};x). \end{aligned}$$

Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| \leq (L_{n,p,q}^{\alpha,\beta}((t-x)^2;x))^{\frac{\rho}{2}}. \tag{3.5}$$

if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L_{n,p,q}^{\alpha,\beta}$ by means of the Peetre's K-functionals.

$C^2[0,1]$: The space of those functions f for which $f, f', f'' \in C[0,1]$. we recall the following norm in the space $C^2[0,1]$:

$$\|f\|_{C^2[0,1]} \ = \ \|f\|_{C[0,1]} \ + \ \|f'\|_{C[0,1]} \ + \ \|f''\|_{C[0,1]}.$$

We consider the following Peetre's K-functional

$$K(f,\delta) := \inf_{g \in C^2[0,1]} \Big\{\|f-g\|_{C[0,1]} \ + \ \delta\|g\|_{C^2[0,1]}\Big\}.$$

$\square$

Theorem 3.3. *Let $f \in C[0,1]$. Then we have*

$$\|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)\|_{C[0,1]} \;\leq\; 2K(f;\delta_n)$$

Where $K(f;\delta_n)$ is Peetre's functional and

$$\begin{aligned}\delta_n \;&= \frac{1}{4}\left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\;+\;\beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q}\;+\;\beta}\right)\\ &+\frac{1}{4}\left(\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right)+\frac{1}{4}\frac{\alpha^2}{([n]_{p,q}+\beta)^2}\;+\;\frac{1}{2}\frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]}.\end{aligned}$$

Proof. Let $g \in C^2[0,1]$. If we use the Taylor's expansion of the function g at $s = x$, we have

$$g(s) \;=\; g(x) \;+\; (s-x)g'(x) \;+\; \frac{(s-x)^2}{2}g''(x).$$

Hence we get

$$\begin{aligned}\|L^{\alpha,\beta}_{n,p,q}(f;x) - f(x)\|_{C[0,1]} \leq&\|L^{\alpha,\beta}_{n,p,q}((s-x);x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}\\ &+\;\frac{1}{2}\|L^{\alpha,\beta}_{n,p,q}((s-x)^2;x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}.\end{aligned} \tag{3.6}$$

From the lemma (2.1) we have

$$\|L^{\alpha,\beta}_{n,p,q}((s-x);x)\|_{C[0,1]} \leq \frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]}. \tag{3.7}$$

So if we use (3.3) and (3.7) in (3.6), then we get

$$\begin{aligned}\|L^{\alpha,\beta}_{n,p,q}(g;x) - g(x)\|_{C[0,1]} \leq&\Bigg[\frac{1}{2}\left(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}\;+\;\beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q}\;+\;\beta}\right)\\ &+\left(\frac{1}{2}\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\right)+\frac{1}{2}\frac{\alpha^2}{([n]_{p,q}+\beta)^2} \qquad (3.8)\\ &+\;\frac{[\alpha]+[\beta]}{[n]+[\beta]}\Bigg]\|g(x)\|_{C[0,1]}.\end{aligned} \tag{3.9}$$

On the other hand, we can write

$$
\begin{aligned}
|L^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\| \ &\le |L^{\alpha,\beta}_{n,p,q}(f-g;x)\| \ + \ |L^{\alpha,\beta}_{n,p,q}(g;x)-g(x)| \\
&+ \ |f(x)-g(x)|.
\end{aligned}
$$

If we take the maximum on $[0,1]$, we have

$$
\|L^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\|_{C[0,1]} \le 2\|f-g\|_{C[0,1]} + \|L^{\alpha,\beta}_{n,p,q}(g;x)-g(x)\|_{C[0,1]}. \quad (3.10)
$$

If we consider (3.8) in (3.10), we obtain

$$
\begin{aligned}
\|L^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\|_{C[0,1]} \ \le & 2\|f-g\|_{C[0,1]} + \Bigg[\frac{1}{4}\Bigg(\frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} \\
& - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\Bigg) + \Bigg(\frac{1}{4}\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\Bigg) \\
& + \frac{1}{4}\frac{\alpha^2}{([n]_{p,q}+\beta)^2} + \ \frac{1}{2}\frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]}\Bigg]\|g(x)\|_{C^2[0,1]}.
\end{aligned}
$$

If we choose

$$
\begin{aligned}
\delta_n \ &= \frac{1}{4}\Bigg(\frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} \ + \ \beta)^2(p(1-x)+qx)} - \frac{[n]_{p,q}-\beta}{[n]_{p,q} \ + \ \beta}\Bigg) \\
&+ \Bigg(\frac{1}{4}\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{([n]_{p,q}+\beta)^2}\Bigg) + \frac{1}{4}\frac{\alpha^2}{([n]_{p,q}+\beta)^2} \ + \ \frac{1}{2}\frac{[\alpha]+[\beta]}{[n]_{p,q}+[\beta]},
\end{aligned}
$$

then we get

$$
\|L^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\|_{C[0,1]} \le 2\Bigg\{\|f-g\|_{C[0,1]} \ + \ \delta_n\|g(x)\|_{C^2[0,1]}\Bigg\}. \quad (3.11)
$$

Finally, one can observe that if we take the infimum of both side of above inequality for the function $g \in C^2[0,1]$, we can find

$$
\|L^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\|_{C[0,1]} \ \le \ 2K(f,\delta_n).
$$

□

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.

Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence $x = (x_k)$ is said to be statistically convergent to a number L if for every $\epsilon > 0$,

$$\delta\{K \in \mathbf{N} : |x_k - L| \geq \varepsilon\} = 0,$$

where $\delta(K)$ is the natural density of the set $K \subseteq \mathbf{N}$.

The density of subset $K \subseteq N$ is defined by

$$\delta(K) := \lim_n \frac{1}{n}\{\text{the number } k \leq n : k \in K\}$$

whenever the limit exists.

For instance, $\delta(\mathbf{N}) = 1$, $\delta\{2K : k \in \mathbf{N}\} = \frac{1}{2}$ and $\delta\{k^2 : K \in \mathbf{N}\} = 0$.

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$X_k = \begin{cases} L_1; \ if \ k = m^2, \\ \\ L_2; \ if \ k \neq m^2. \end{cases} \quad where \ m \in \mathbf{N} \tag{4.1}$$

is statistically convergent to L_2 but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statiscally convergent but not conversley.

Now we consider sequences $q = q_n$ and $p = p_n$ such that:

$$st - \lim_n q_n = 1, \ st - \lim_n p_n = 1, \text{ and } st - \lim_n q_n^n = 1. \tag{4.2}$$

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.

Theorem 4.1. *If A_n be the sequence of linear positive operators from $C[a,b]$ to $C[a,b]$ satisfies the conditions*
$st - \lim_n \|A_n((t^\nu;x)) - (x)^\nu\|_C[0,1] = 0$ *for* $\nu = 0,\ 1,\ 2.$

then for any function $f \in C[a,b]$,

$st - \lim_n \|A_n(f;.) - f\|_C[a,b] = 0.$

Now we will discuss the rates of statistical convergence of $L^{\alpha,\beta}_{n,p,q}$ operators.

Remark 4.2. *For $q \in (0,1)$ and $p \in (q,1]$, it is obvious that*

$$\lim_{n\to\infty}[n]_{p,q} = \begin{cases} 0, \ \textit{when} \ \ p,q \in (0,1) \\ \\ \frac{1}{1-q}, \quad \textit{when } p = 1 \textit{ and } q \in (0,1). \end{cases}$$

In order to reach to convergence results of the operator $L^n_{p,q}(f;x)$, we take a sequence $q_n \in (0,1)$ and $p_n \in (q_n,1]$ such that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. So we get $\lim_{n\to\infty}[n]_{p_n,q_n} = \infty$.

Theorem 4.3. *Let $L^{\alpha,\beta}_{n,p,q}$ be the sequence of operators and the sequences $p = p_n$ and $q = q_n$ satisfies Remark 4.2 then for any function $f \in C[0,1]$*

$$st - \lim_n \|\ L^{\alpha,\beta}_{n,p_n,q_n}(f,.) - f\| = 0. \tag{4.3}$$

Proof. Clearly for $\nu = 0$,

$$L^{\alpha,\beta}_{n,p_n,q_n}(1,x) = 1,$$

which implies

$$st - \lim_n \|L^{\alpha,\beta}_{n,p_n,q_n}(1;x)\ - 1\ \| \ = \ 0.$$

For $\nu \ = \ 1$

$$\begin{aligned}\|L^{\alpha,\beta}_{n,p_n,q_n}\ (t;x)\ -\ x\ \| &\le \left|\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}x \ + \ \frac{\alpha}{[n]_{p_n,q_n}+\beta}\ -\ x\right| \\ &= \left|\left(\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}\ -\ 1\right)x \ + \ \frac{\alpha}{[n]_{p_n,q_n}+\beta}\right| \\ &\le \left|\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}\ -\ 1\right| \ + \ \left|\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right|.\end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets.

$$U = \{n : \|L^{\alpha,\beta}_{n,p_n,q_n}(t;x) - x\| \geq \epsilon\}$$
$$U' = \{n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta}\} \geq \epsilon$$
$$U'' = \{n : \frac{\alpha}{[n]_{p_n,q_n} + \beta} \geq \epsilon\}.$$

It is obvious that $U \subseteq U'' \cup U'$,

So using

$$\delta\{k \leq n : 1 - \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} \ \geq \epsilon\},$$

then we get

$$st - \lim_n \|L^{\alpha,\beta}_{n,p_n,q_n}(t;x) - x\| = 0. \tag{4.4}$$

Lastly for $\nu = 2$, we have

$$\|L^{\alpha,\beta}_{n,p_n,q_n}(t^2 : x) - x^2\| \leq \Big|\frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x) + qx}\frac{1}{\left([n]_{p_n,q_n} + \beta\right)^2} \ - 1\Big|$$
$$+ \Big|\frac{[n]_{p_n,q_n}\left(2\alpha + p^{n-1}\right)^2}{[n]_{p_n,q_n} + \beta} x\Big| + \Big|\frac{\alpha^2}{\left([n]_{p_n,q_n} + \beta\right)^2}\Big|.$$

If we choose

$$\alpha_n = \frac{q^2[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{p(1-x) + qx}\frac{1}{\left([n]_{p_n,q_n} + \beta\right)^2} \ - 1$$
$$\beta_n = \frac{[n]_{p_n,q_n}\left(2\alpha + p^{n-1}\right)^2}{[n]_{p_n,q_n} + \beta}$$
$$\gamma_n = \frac{\alpha^2}{\left([n]_{p_n,q_n} + \beta\right)^2}.$$

Then

$$st - \lim_n \alpha_n \ = \ st - \lim_n \beta_n \ = \ st - \lim_n \gamma_n \ = \ 0.$$

Now given $\epsilon > 0$, we define the following four sets:

$$U \ = \ \|L^{\alpha,\beta}_{n,p_n,q_n}(t^2 : x) - x^2\| \geq \epsilon,$$
$$U_1 = \{n : \alpha_n \geq \frac{\epsilon}{3}\},$$
$$U_2 = \{n : \beta_n \geq \frac{\epsilon}{3}\},$$

$$U_3 = \{n : \gamma_n \geq \frac{\epsilon}{3}\}.$$

It is obvious that $U \subseteq U_1 \bigcup U_2 \bigcup U_3$. Thus we obtain

$$\delta\{K \leq n : \|L^{\alpha,\beta}_{n,p,q}(t^2 : x) - x^2\| \geq \epsilon\}$$
$$\leq \delta\{K \leq n : \alpha_n \geq \frac{\epsilon}{3}\} \ + \ \delta\{K \leq n : \beta_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \gamma_n \geq \frac{\epsilon}{3}\}.$$

So the right hand side of the inequalities is zero.

Then

$$st - \lim_n \|L^{\alpha,\beta}_{n,p_n,q_n}(t; x) - x\| = 0$$

holds and thus the proof is completed.

□

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