Approximation by $(p, q)$-Lupaş Stancu Operators

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Abstract. In this paper, $(p, q)$-Lupaş Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of $(p, q)$-Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: $(p, q)$-Integers, Lupaş $(p, q)$-Bernstein Stancu operators, Statistical approximation, Korovkin’s type approximation.


1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators $B_n : C[0, 1] \rightarrow C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad x \in [0, 1].$$

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].

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Further, based on $q$-integers, Lupas [21] introduced the first $q$-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another $q$-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on $q$-integers have been introduced and their approximation properties studied.

Recently, the applications of $(p,q)$-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving $(p,q)$-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen et al [27] introduced the concept of post quantum calculus in approximation theory and constructed the $(p,q)$-analogue of Bernstein operators defined as follows for $0 < q < p \leq 1$:

$$B_{n,p,q}(f;x) = \frac{1}{p^{n(n+1)} x} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} \frac{k(k-1)}{2} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f \left( \frac{[k]_{p,q}}{p^n - [n]_{p,q}} \right), \quad x \in [0,1].$$

(1.2)

Note when $p = 1$, $(p,q)$-Bernstein Operators given by (1.2) turns out to be Phillips $q$-Bernstein Operators [38].

Also, we have

$$(1-x)^n_{p,q} = \prod_{s=0}^{n-1} (p^s - q^s x) = (1-x)(p-x)(p^2 - q^2 x)...(p^{n-1} - q^{n-1} x)$$

$$= \sum_{k=0}^{n} (-1)^k p^{(n-k)(n-k-1)} q^{k(k-1)} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k.$$

Further, they applied the concept of $(p,q)$-calculus in approximation theory and studied approximation properties based on $(p,q)$-integers for Bernstein-Stancu operators, $(p,q)$-analogue of Bernstein-Kantorovich, $(p,q)$-analogue of Bernstein-Shurer operators, $(p,q)$-analogue of Bleimann-Butzer-Hahn operators and $(p,q)$-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined $(p,q)$-analogue of Lupas Bernstein operators [17] as follows:
For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0, 1] \rightarrow C[0, 1]$ as

$$L^n_{p,q}(f; x) = \frac{f\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}}\right)}{\prod_{j=1}^{n} \left\{p^{j-1}(1-x) + q^{j-1}x\right\}} \sum_{k=0}^{n} f\left(p^{n-k} [k]_{p,q}\left[n\right]_{p,q}\left[p\right](n-k)\left[q\right](n-k-1)\right) x^k (1-x)^{n-k},$$

are $(p,q)$-analogue of Lupaş Bernstein operators.

Again when $p = 1$, Lupaş $(p,q)$-Bernstein operators turns out to be Lupaş $q$-Bernstein operators as given in [22, 37].

When $p = q = 1$, Lupaş $(p,q)$-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau’s algorithm and Korovkin’s type approximation properties [17]: de-Casteljau’s algorithm and related results of degree elevation reduction for Bézier curves and surfaces holds for all $p > 0$ and $q > 0$. However to study Korovkin’s type approximation properties for Lupaş $(p,q)$-analogue of the Bernstein operators, $0 < q < p \leq 1$ is needed.

Based on Korovkin’s type approximation, they proved that the sequence of $(p,q)$-analogue of Lupaş Bernstein operators $L^n_{p,q,n}(f, x)$ converges uniformly to $f(x) \in C[0, 1]$ if and only if $0 < q_n < p_n \leq 1$ such that $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} p_n = 1$ and $\lim_{n \to \infty} q_n^n = 1$. On the other hand, for any $p > 0$ fixed and $p \neq 1$, the sequence $L^n_{p,q}(f, x)$ converges uniformly to $f(x) \in C[0, 1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Furthermore, in comparison to $q$-Bézier curves and surfaces based on Lupaş $q$-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter $p$ have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of $(p,q)$-calculus.

For any $p > 0$ and $q > 0$, the $(p,q)$ integers $[n]_{p,q}$ are defined by
\[ [n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q}, & \text{when } p \neq q \\ n & \text{when } p = q \\ [n], & \text{when } p = q = 1 \end{cases} \]

where \([n]_q\) denotes the \(q\)-integers and \(n = 0, 1, 2, \ldots\).

The formula for \((p, q)\)-binomial expansion is as follows:

\[ (ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} a^{n-k}b^k x^{n-k}y^k, \]

\[ (x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y), \]

\[ (1 - x)^n_{p,q} = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x), \]

where \((p, q)\)-binomial coefficients are defined by

\[ \binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}. \]

Details on \((p, q)\)-calculus can be found in [10, 11, 27].

Also, we have \((p, q)\)-analogue of Euler’s identity as:

\[ (1 - x)^n_{p,q} = \prod_{s=0}^{n-1} (p^s - q^s x) = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x) \]

\[ = \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} x^k. \]

Again by some simple calculations and using the property of \((p, q)\)-integers, we get \((p, q)\)-analogue of Pascal’s relation as follows:

\[ \begin{align*}
\binom{n}{k}_{p,q} & = q^{n-k} \binom{n-1}{k-1}_{p,q} + p^k \binom{n-1}{k}_{p,q} \\
\binom{n}{k}_{p,q} & = p^{n-k} \binom{n-1}{k-1}_{p,q} + q^k \binom{n-1}{k}_{p,q} .
\end{align*} \]

(1.4)

(1.5)

We recall some results from [17] for Lupas \((p, q)\)-Bernstein operators, which reproduces linear and constant functions.
Some auxiliary results:

(1) \( L_n^{n}(1, \frac{u}{u+1}) = 1 \)

(2) \( L_n^{n}(t, \frac{u}{u+1}) = \frac{u}{u+1} \)

(3) \( L_n^{n}(t^2, \frac{u}{u+1}) = \frac{u}{u+1} \left( \frac{u^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} \left( \frac{[n-1]_{p,q}}{[n]_{p,q}} \right) \right) \)

or equivalently for \( x = \frac{u}{u+1} \)

\( L_n^{n}(1, x) = 1 \), \hspace{1cm} (1.6)

\( L_n^{n}(t, x) = x \), \hspace{1cm} (1.7)

\( L_n^{n}(t^2; x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2x^2}{p(1-x) + qx} \left[ \frac{[n-1]_{p,q}}{[n]_{p,q}} \right] \), \hspace{1cm} (1.8)

2. Construction of \((p, q)\)-Lupaş Stancu Operators

In this section, we introduce \((p, q)\)-Lupaş Stancu operators as follows:

For any \( p > 0 \) and \( q > 0 \), the linear operators \( L_n^{n,p,q} : C[0, 1] \rightarrow C[0, 1] \)

\[
L_{n,p,q}^{\alpha,\beta}(f; x) = \sum_{k=0}^{n} f \left( \frac{P^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{k,n}^{p,q}(t)
\] \hspace{1cm} (2.1)

and \( b_{k,n}^{p,q}(t) \) is given by

\[
b_{k,n}^{p,q}(t) = \frac{\binom{n}{k}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^k \left( 1-t \right)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}},
\] \hspace{1cm} (2.2)

where \( 0 < \alpha < \beta \).

We give some equalities for operators (2.1) in the following lemma.

**Lemma 4.1.** The following equalities are true:

(i) \( L_{n,p,q}^{\alpha,\beta}(1; x) = 1 \),

(ii) \( L_{n,p,q}^{\alpha,\beta}(t; x) = \frac{[n]_{p,q}x + \alpha}{[n]_{p,q} + \beta} \),

(iii) \( L_{n,p,q}^{\alpha,\beta}(t^2; x) = \frac{[n]_{p,q}x^2 [n-1]_{p,q}}{p(1-x) + qx} x^2 + \frac{[n]_{p,q}(2\alpha + p^{n-1})}{((n)_{p,q} + \beta)^2} x + \frac{\alpha^2}{(n)_{p,q} + \beta}. \)
Proof. Proof of part (i) is obvious.

\[
L_{\alpha,\beta}^{n,p,q}(t; x) = \sum_{k=0}^{n} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{k,n}^{p,q}(t)
\]

\[
= \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(t; x) + \frac{[\alpha]}{[n]_{p,q} + [\beta]} L_{p,q}^{n}(1; x).
\]

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

\[
L_{\alpha,\beta}^{n,p,q}(t^2; x) = \sum_{k=0}^{n} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{k,n}^{p,q}(t)
\]

\[
= \frac{1}{([n]_{p,q} + \beta)^2} \left[ p^{2n-2k} [k]_{p,q}^2 b_{k,n}^{p,q}(t) + 2\alpha p^{n-k} [k]_{p,q} b_{k,n}^{p,q}(t) + \alpha^2 b_{k,n}^{p,q}(t) \right]
\]

\[
= \frac{1}{([n]_{p,q} + \beta)^2} \left[ A + B + C \right].
\]

\[
A = \sum_{k=0}^{n} \frac{[k]^2}{p^{2k}} \prod_{j=1}^{n} \left\{ \frac{p^{j-1}(1-t) + q^{j-1}t}{p^{j-1} + q^{j-1}u} \right\}
\]

On shifting the limits and on replacing \( k \) by \( k + 1 \), we get

\[
A = \sum_{k=1}^{n} \frac{[k+1]}{p^{2k+2}} \prod_{j=1}^{n-1} \left\{ \frac{p^j + q^j u}{p^j + q^j u} \right\}
\]

\[
= \sum_{k=0}^{n-1} \frac{[k+1]}{p^{k+2}} \prod_{j=0}^{n-2} \left\{ \frac{p^j + q^j u}{p^j + q^j u} \right\}
\]

\[
= \sum_{k=0}^{n-1} \frac{[k+1]}{p^{k+2}} \prod_{j=0}^{n-2} \left\{ \frac{p^j + q^j u}{p^j + q^j u} \right\}
\]
Using \([k + 1]_{p,q} = p^k + q[k]_{p,q}\), we get our desired result:

\[
A = [n]_{p,q} \frac{u}{u + 1} \sum_{k=0}^{n-1} \frac{p^k + q[k]}{p^{k+2}} \frac{\left\{ n-1 \atop k \right\}_{p,q}}{p^{n-2} \prod_{j=0}^{n-2} \left\{ p^j + q^j \left( \frac{q}{p} \right)^j \right\}},
\]

\[
= [n]_{p,q} p^{n-1} \frac{u}{u + 1} + \frac{q^2 u^2 [n]_{p,q}[n-1]_{p,q}}{(u + 1)(p + qa)},
\]
equivalently

\[
A = [n]_{p,q} p^{n-1}x + \frac{q^2 [n]_{p,q}[n-1]_{p,q} x^2}{(p(1-x) + qx)}.
\]

Similarly

\[
B = 2\alpha [n] p^n \sum_{k=0}^{n} \frac{\left\{ k \atop n \right\}_{p,q}}{p^k} \frac{p^{(n-k)(n-k-1)}}{q^{\frac{k(k-1)}{2}}} \frac{q^{\frac{k(k-1)}{2}}}{t^k (1-t)^{n-k}} \prod_{j=1}^{n} \left\{ p^j + q^j (\frac{at}{p}) \right\},
\]

\[
= 2\alpha [n] p^n \frac{1}{p^k} \sum_{k=1}^{n} \frac{\left\{ n-1 \atop k-1 \right\}_{p,q}}{p^{n-2} \prod_{j=1}^{n-1} \left\{ p^j + q^j (\frac{at}{p}) \right\}}
\]

After shifting the limits and on replacing \(k\) by \(k + 1\), we get

\[
B = 2\alpha [n] p^n \frac{u}{u + 1} \sum_{k=0}^{n-1} \frac{1}{p^k} \frac{\left\{ n-1 \atop k \right\}_{p,q}}{p^{n-2} \prod_{j=0}^{n-2} \left\{ p^j + q^j (\frac{at}{p}) \right\}},
\]

which implies

\[
B = 2\alpha [n]_{p,q} x.
\]

Similarly

\[
C = \alpha^2 \sum_{k=0}^{n} \frac{\left\{ n \atop k \right\}_{p,q}}{p^{n-k} \prod_{j=1}^{n-k} \left\{ p^j + q^j (\frac{at}{p}) \right\}}
\]

\[
= \alpha^2.
\]
Theorem 2.1. Let $0 < q_n < p_n \leq 1$ such that $\lim_{n \to \infty} p_n = 1$, $\lim_{n \to \infty} q_n = 1$, and for $f \in C[0,1]$, we have $\lim_n |L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)| = 0$.

Proof. Let us recall the following Korovkin’s theorem see [20]. Let $(T_n)$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_{C[a,b]} = 0$, for $i = 0, 1, 2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators $L_{n,p,q}^{\alpha,\beta}(f;x)$ to the functions $f$ by means of modulus of continuity, elements of Lipschitz class and peetre’s K-functional.

Let $f \in C[0,1]$. The modulus of continuity of $f$ denoted by $\omega(f, \delta)$ is defined as:

$$\omega(f, \delta) = \sup_{y,x \in [0,1], |y-x| < \delta} |f(y) - f(x)|.$$

where $w(f;\delta)$ satisfies the following conditions: for all $f \in C[0,1]$,

$$\lim_{\delta \to 0} w(f;\delta) = 0.$$  

and

$$|f(y) - f(x)| \leq w(f; \delta) \left(\frac{|y - x|}{\delta} + 1\right).$$

Theorem 3.1. Let $0 < q < p \leq 1$, and $f \in C[0,1]$, and $\delta > 0$, we have

$$\|L_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,1]} \leq 2\omega(f;\delta_n)$$

where

$$\delta_n = \left[\left(\frac{q^2[n]_{p,q}(n-1)_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2}\right) + \left(\frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2}\right)^{\frac{1}{2}}\right].$$
Proof. From lemma (4.1) we have

\[
|L_{\alpha,\beta}^{n,p,q}(t - x)^2; x) = \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1 - x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right)x^2
+ \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right)x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.
\]

(3.3)

For \(x \in [0,1]\), we take

\[
|L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)| \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta}(L_{\alpha,\beta}^{n,p,q}(t - x)^2; x)^{\frac{3}{2}} \right\},
\]

then we get

\[
\|L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)\|_{C[0,1]} \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left( \frac{1}{([n]_{p,q} + \beta)^2} \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1 - x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \right) \right. \\
+ \left. \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right\}.
\]

If we choose

\[
\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1 - x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \right. \\
+ \left. \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}.
\]

Then we have

\[
\|L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n).
\]

So we have the desired result. □

Now we compute the approximation order of operator \(L_{\alpha,\beta}^{n,p,q}\) in term of the elements of the usual Lipschitz class.
Let $f \in C[0, 1]$ and $0 < \rho \leq 1$. We recall that $f$ belongs to $\text{Lip}_M(\rho)$ if the inequality
\[ |f(x) - f(y)| \leq M|x - y|^\rho; \text{ for all } x, y \in [0, 1] \] (3.4)
holds.

**Theorem 3.2.** For all $f \in \text{Lip}_M(\rho)$
\[ \|L^{\alpha,\beta}_{n,p,q}(f; x) - f(x)\|_{C[0,1]} \leq M\delta_n^\rho \]
where
\[ \delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2} p(1-x) + qx \right) - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right]^\frac{1}{2} \]
and $M$ is a positive constant.

**Proof.** Let $f \in \text{Lip}_M(\rho)$ and $0 < \rho \leq 1$. by (3.4) and linearity and monotonicity of $L^{\alpha,\beta}_{n,p,q}$ then we have
\[ |L^{\alpha,\beta}_{n,p,q}(f; x) - f(x)| \leq L^{\alpha,\beta}_{n,p,q}(|f(t) - f(x)|; x) \]
\[ \leq L^{\alpha,\beta}_{n,p,q}(|t - x|^\rho; x). \]
Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get
\[ |L^{\alpha,\beta}_{n,p,q}(f; x) - f(x)| \leq (L^{\alpha,\beta}_{n,p,q}((t - x)^2; x))^\frac{1}{2}. \] (3.5)
if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L^{\alpha,\beta}_{n,p,q}$ by means of the Peetre’s K-functionals.

$C^2[0,1]$: The space of those functions $f$ for which $f, f', f'' \in C[0,1]$. we recall the following norm in the space $C^2[0,1]$
\[ \|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}. \]
We consider the following Peetre’s K-functional
\[ K(f, \delta) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\|_{C[0,1]} + \delta\|g\|_{C^2[0,1]} \right\}. \]
Theorem 3.3. Let \( f \in C[0, 1] \). Then we have
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0, 1]} \leq 2K(f; \delta_n)
\]
where \( K(f; \delta_n) \) is Peetre’s functional and
\[
\delta_n = \frac{1}{4} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \frac{1}{4} \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \left( \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.
\]

Proof. Let \( g \in C^2[0, 1] \). If we use the Taylor’s expansion of the function \( g \) at \( s = x \), we have
\[
g(s) = g(x) + (s-x)g'(x) + \frac{(s-x)^2}{2}g''(x).
\]
Hence we get
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0, 1]} \leq \|L_{n,p,q}^{\alpha,\beta}((s-x); x)\|_{C[0, 1]}\|g(x)\|_{C^2[0, 1]}
\]
\[
+ \|L_{n,p,q}^{\alpha,\beta}((s-x)^2); x\|_{C[0, 1]}\|g(x)\|_{C^2[0, 1]}.
\]
(3.6)

From the lemma (2.1) we have
\[
\|L_{n,p,q}^{\alpha,\beta}((s-x); x)\|_{C[0, 1]} \leq \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.
\]
(3.7)

So if we use (3.3) and (3.7) in (3.6), then we get
\[
\|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0, 1]} \leq \frac{1}{2} \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \frac{1}{2} \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{2} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}\|g(x)\|_{C[0, 1]}.
\]
(3.8)

On the other hand, we can write
\[ |L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq |L_{n,p,q}^{\alpha,\beta}(f - g; x)| + |L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)| + |f(x) - g(x)|. \]

If we take the maximum on \([0, 1]\), we have

\[ \|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]} + \|f(x) - g(x)\|. \] (3.10)

If we consider (3.8) in (3.10), we obtain

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \left[ \frac{1}{4} \frac{q[n]_{p,q} [n - 1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1 - x) + qx)} \right] \\
- \left[ \frac{[n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2} \right] + \left[ \frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right] \\
+ \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]} \|g(x)\|_{C^2[0,1]}.
\]

If we choose

\[
\delta_n = \frac{1}{4} \left( \frac{q^2[n]_{p,q} [n - 1]_{p,q}}{([n]_{p,q} + \beta)^2 (p(1 - x) + qx)} - \frac{[n]_{p,q} - \beta}{([n]_{p,q} + \beta)^2} \right) \\
+ \left( \frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \alpha^2 + \frac{1}{2} \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]},
\]

then we get

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2 \left\{ \|f - g\|_{C[0,1]} + \delta_n \|g(x)\|_{C^2[0,1]} \right\}. \] (3.11)

Finally, one can observe that if we take the infimum of both side of above inequality for the function \(g \in C^2[0,1]\), we can find

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f, \delta_n).
\]

\[
□
\]

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.
Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence \( x = (x_k) \) is said to be statistically convergent to a number \( L \) if for every \( \varepsilon > 0 \),

\[
\delta \{ K \in \mathbb{N} : |x_k - L| \geq \varepsilon \} = 0,
\]

where \( \delta(K) \) is the natural density of the set \( K \subseteq \mathbb{N} \).

The density of subset \( K \subseteq \mathbb{N} \) is defined by

\[
\delta(K) := \lim \frac{1}{n} \{ \text{the number } k \leq n : k \in K \}
\]

whenever the limit exists.

For instance, \( \delta(\mathbb{N}) = 1 \), \( \delta\{2K : k \in \mathbb{N}\} = \frac{1}{2} \) and \( \delta\{k^2 : K \subseteq \mathbb{N}\} = 0 \).

To emphasize the importance of the statistical convergence, we have an example: The sequence

\[
X_k = \begin{cases} 
L_1; & \text{if } k = m^2, \\
L_2; & \text{if } k \neq m^2.
\end{cases}
\]

where \( m \in \mathbb{N} \) (4.1)

is statistically convergent to \( L_2 \) but not convergent in ordinary sense when \( L_1 \neq L_2 \). We note that any convergent sequence is statistically convergent but not conversely.

Now we consider sequences \( q = q_n \) and \( p = p_n \) such that:

\[
st - \lim \frac{q_n}{n} = 1, \ st - \lim \frac{p_n}{n} = 1, \ \text{and} \ st - \lim \frac{q_n^a}{n} = 1.
\]

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.
Theorem 4.1. If \( A_n \) be the sequence of linear positive operators from \( C[a, b] \) to \( C[a, b] \) satisfies the conditions
\[
st - \lim_{n} \|A_n((t^\nu;x)) - (x^\nu)\|_{C[0,1]} = 0 \quad \text{for } \nu = 0, 1, 2.
\]

then for any function \( f \in C[a, b] \),
\[
st - \lim_{n} \|A_n(f;) - f\|_{C[a,b]} = 0.
\]

Now we will discuss the rates of statistical convergence of \( L_{\alpha,\beta}^{n,p,q} \) operators.

Remark 4.2. For \( q \in (0,1) \) and \( p \in (q,1) \), it is obvious that
\[
\lim_{n \to \infty} [n]_{p,q} = \begin{cases} 
0, & \text{when } p, q \in (0,1) \\
\frac{1}{1-q}, & \text{when } p = 1 \text{ and } q \in (0,1).
\end{cases}
\]

In order to reach to convergence results of the operator \( L_{\alpha,\beta}^{n,p,q}(f;x) \), we take a sequence \( q_n \in (0,1) \) and \( p_n \in (q_n,1) \) such that \( \lim_{n \to \infty} p_n = 1, \lim_{n \to \infty} q_n = 1 \). So we get \( \lim_{n \to \infty} [n]_{p_n,q_n} = \infty \).

Theorem 4.3. Let \( L_{\alpha,\beta}^{\alpha,\beta} \) be the sequence of operators and the sequences \( p = p_n \) and \( q = q_n \) satisfies Remark 4.2 then for any function \( f \in C[0,1] \)
\[
st - \lim_{n} \|L_{\alpha,\beta}^{\alpha,\beta}(f;) - f\| = 0. \quad (4.3)
\]

Proof. Clearly for \( \nu = 0 \),
\[
L_{\alpha,\beta}^{\alpha,\beta}(1;x) = 1,
\]
which implies
\[
st - \lim_{n} \|L_{\alpha,\beta}^{\alpha,\beta}(1;x) - 1\| = 0.
\]

For \( \nu = 1 \)
\[
\|L_{\alpha,\beta}^{\alpha,\beta}(t;x) - x\| \leq \left| \frac{[n]_{p_n,q_n} x}{[n]_{p_n,q_n} + \beta} - x \right| + \frac{\alpha}{[n]_{p_n,q_n} + \beta} - x \]
\[
= \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1 \right| x + \frac{\alpha}{[n]_{p_n,q_n} + \beta} - x \]
\[
\leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1 \right| + \frac{\alpha}{[n]_{p_n,q_n} + \beta}.
\]
For a given \( \epsilon > 0 \), let us define the following sets.

\[
U = \{ n : \| L_{n,p,q}^{\alpha,\beta}(t;x) - x \| \geq \epsilon \}
\]
\[
U' = \{ n : 1 - \frac{[n]_{p,q}}{[n]_{p,q} + \beta} \geq \epsilon \}
\]
\[
U'' = \{ n : \frac{\alpha}{[n]_{p,q} + \beta} \geq \epsilon \}.
\]

It is obvious that \( U \subseteq U'' \cup U' \).

So using

\[
\delta \{ k \leq n : 1 - \frac{[n]_{p,q}}{[n]_{p,q} + \beta} \geq \epsilon \},
\]

then we get

\[
\text{st} \lim_n \| L_{n,p,q}^{\alpha,\beta}(t;x) - x \| = 0.
\]

(4.4)

Lastly for \( \nu = 2 \), we have

\[
\| L_{n,p,q}^{\alpha,\beta}(t^2 : x) - x^2 \| \leq \left| \frac{q^2[n]_{p,q} [n-1]_{p,q}}{p(1-x) + qx} \left( \frac{1}{[n]_{p,q} + \beta} \right)^2 - 1 \right|
\]
\[
+ \left| \frac{[n]_{p,q}}{[n]_{p,q} + \beta} \frac{(2\alpha + p^{\nu-1})^2}{x} + \left( \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right)^2 \right|.
\]

If we choose

\[
\alpha_n = \frac{q^2[n]_{p,q} [n-1]_{p,q}}{p(1-x) + qx} \left( \frac{1}{[n]_{p,q} + \beta} \right)^2 - 1
\]
\[
\beta_n = \frac{[n]_{p,q} (2\alpha + p^{\nu-1})^2}{[n]_{p,q} + \beta}
\]
\[
\gamma_n = \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.
\]

Then

\[
\text{st} \lim_n \alpha_n = \text{st} \lim_n \beta_n = \text{st} \lim_n \gamma_n = 0.
\]

Now given \( \epsilon > 0 \), we define the following four sets:

\[
U = \| L_{n,p,q}^{\alpha,\beta}(t^2 : x) - x^2 \| \geq \epsilon,
\]
\[
U_1 = \{ n : \alpha_n \geq \frac{\epsilon}{3} \},
\]
\[
U_2 = \{ n : \beta_n \geq \frac{\epsilon}{3} \},
\]
It is obvious that $U \subseteq U_1 \cup U_2 \cup U_3$. Thus we obtain

\[
\delta\{K \leq n : \|L^{\alpha,\beta}_{n,p,q}(t^2 : x) - x^2\| \geq \epsilon\} \\
\leq \delta\{K \leq n : \alpha_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \beta_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \gamma_n \geq \frac{\epsilon}{3}\}.
\]

So the right hand side of the inequalities is zero.

Then

\[
st - \lim_{n} \|L^{\alpha,\beta}_{n,p,n,q_n}(t : x) - x\| = 0
\]

holds and thus the proof is completed.

\[\square\]

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REFERENCES


42. A. Wafi, N. Rao, Bivariate-Schurer-Stancu operators based on \((p, q)\)-integers, Filomat, 32(4), (2018), 1251-1258.
45. A. Wafi, N. Rao, A generalization of Szasz-type operators which preserves constant and quadratic test functions, cogent mathematics, 3(1), 2016, 1227023.