Approximation by \((p, q)\)-Lupaş Stancu Operators

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Abstract. In this paper, \((p, q)\)-Lupaş Bernstein Stancu operators are constructed. Statistical as well as other approximation properties of \((p, q)\)-Lupaş Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated.

Keywords: \((p, q)\)-Integers, Lupaş \((p, q)\)-Bernstein Stancu operators, Statistical approximation, Korovkin’s type approximation.


1. Introduction and Preliminaries

In 1912, S.N. Bernstein [6] introduced his famous operators \(B_n : C[0, 1] \to C[0, 1]\) defined for any \(n \in \mathbb{N}\) and for any function \(f \in C[0, 1]\)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1].
\] (1.1)

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [20].
Further, based on $q$-integers, Lupaş [21] introduced the first $q$-Bernstein operators [6] and investigated its approximating and shape-preserving properties. Another $q$-analogue of the Bernstein polynomials is due to Phillips [38]. Since then several generalizations of well-known positive linear operators based on $q$-integers have been introduced and their approximation properties studied.

Recently, the applications of $(p,q)$-calculus (post quantum calculus) emerged as a new area in the field of approximation theory [20]. The development of post quantum calculus has led to the discovery of various generalizations of Bernstein polynomials involving $(p,q)$-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design [7] and solutions of differential equations.

Mursaleen et al [27] introduced the concept of post quantum calculus in approximation theory and constructed the $(p,q)$-analogue of Bernstein operators defined as follows for $0 < q < p \leq 1$:

$$B_{n,p,q}(f; x) = \frac{1}{p^{-n(n-1)}} \sum_{k=0}^{n} \binom{n}{k}_{p,q} \frac{1}{p^{k(k-1)}x^k} \prod_{s=0}^{n-k-1} (p^s - q^s x^s) f \left( \frac{[k]_{p,q}}{p^n - [n]_{p,q}} \right), \ x \in [0, 1].$$

(1.2)

Note when $p = 1$, $(p,q)$-Bernstein Operators given by (1.2) turns out to be Phillips $q$-Bernstein Operators [38].

Also, we have

$$(1 - x)^n_{p,q} = \prod_{k=0}^{n-1} (p^k - q^k x) = (1 - x)(p - qx)(p^2 - q^2x) \ldots (p^{n-1} - q^{n-1}x)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}_{p,q} \frac{(n-k)(n-k-1)}{x^k} \prod_{s=0}^{k-1} (p^s - q^s x^s).$$

Further, they applied the concept of $(p,q)$-calculus in approximation theory and studied approximation properties based on $(p,q)$-integers for Bernstein-Štancu operators, $(p,q)$-analogue of Bernstein-Kantorovich, $(p,q)$-analogue of Bernstein-Shurer operators, $(p,q)$-analogue of Bleimann-Butzer-Hahn operators and $(p,q)$-analogue of Lorentz polynomials on a compact disk in [28, 31, 32, 33, 35].

On the other hand, Khalid and Lobiyal defined $(p,q)$-analogue of Lupaş Bernstein operators [17] as follows:
For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0, 1] \to C[0, 1]$ as

\[
L^n_{p,q}(f; x) = \sum_{k=0}^{n} f\left(\frac{p^{n-k} [k]_{p,q}}{[n]_{p,q}} \right) p^{\frac{(n-k)(q-k-1)}{2}} q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k} \prod_{j=1}^{n} \left(p^{j-1}(1-x) + q^{j-1}x\right),
\]

(1.3)

are $(p, q)$-analogue of Lupaş Bernstein operators.

Again when $p = 1$, Lupaş $(p,q)$-Bernstein operators turns out to be Lupaş $q$-Bernstein operators as given in [22, 37].

When $p = q = 1$, Lupaş $(p,q)$-Bernstein operators turns out to be classical Bernstein operators [6].

They studied two different techniques as de-Casteljau’s algorithm and Korovkin’s type approximation properties [17]: de-Casteljau’s algorithm and related results of degree elevation reduction for Bézier curves and surfaces holds for all $p > 0$ and $q > 0$. However to study Korovkin’s type approximation properties for Lupaş $(p,q)$-analogue of the Bernstein operators, $0 < q < p \leq 1$ is needed.

Based on Korovkin’s type approximation, they proved that the sequence of $(p,q)$-analogue of Lupaş Bernstein operators $L^n_{p,q,n}(f; x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $0 < q_n < p_n \leq 1$ such that $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} p_n = 1$ and $\lim_{n \to \infty} q^n = 1$. On the other hand, for any $p > 0$ fixed and $p \neq 1$, the sequence $L^n_{p,q}(f; x)$ converges uniformly to $f(x) \in C[0,1]$ if and only if $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Furthermore, in comparison to $q$-Bézier curves and surfaces based on Lupaş $q$-Bernstein rational functions, their generalization gives more flexibility in controlling the shapes of curves and surfaces.

Some advantages of using the extra parameter $p$ have been discussed in the field of approximations on compact disk [35] and in computer aided geometric design [17].

For more details related to approximation theory [20], one can refer [1, 2, 3, 5, 8, 9, 12, 13, 14, 15, 18, 19, 22, 23, 24, 34, 36, 39, 40, 42, 43, 44, 45, 46, 47, 48].

Let us recall certain notations of $(p,q)$-calculus.

For any $p > 0$ and $q > 0$, the $(p,q)$ integers $[n]_{p,q}$ are defined by
\[ [n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \ldots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, & \text{when } p \neq q \neq 1 \\ n p^{n-1}, & \text{when } p = q \neq 1 \\ [n]_q, & \text{when } p = 1 \\ n, & \text{when } p = q = 1 \end{cases} \]

where \([n]_q\) denotes the \(q\)-integers and \(n = 0, 1, 2, \ldots\).

The formula for \((p, q)\)-binomial expansion is as follows:

\[
(ax + by)_{p,q}^n := \sum_{k=0}^{n} \binom{n}{k}_{p,q} \frac{(n-k)(n-k-1)}{q^{k(k-1)}} \binom{n}{k}_{p,q} a^{n-k}b^k x^{n-k} y^k,
\]

\[
(x+y)_{p,q}^n = (x+y)(px + qy)(p^2x + q^2y) \ldots (p^{n-1}x + q^{n-1}y),
\]

\[
(1-x)_{p,q}^n = (1-x)(p - qx)(p^2 - q^2x) \ldots (p^{n-1} - q^{n-1}x),
\]

where \((p, q)\)-binomial coefficients are defined by

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.
\]

Details on \((p, q)\)-calculus can be found in [10, 11, 27].

Also, we have \((p, q)\)-analogue of Euler’s identity as:

\[
(1-x)_{p,q}^n = \prod_{s=0}^{n-1} (p^s - q^s x) = (1-x)(p - qx)(p^2 - q^2x) \ldots (p^{n-1} - q^{n-1}x)
\]

\[
= \sum_{k=0}^{n} (-1)^k p^\frac{(n-k)(n-k-1)}{2} q^{\frac{k(k-1)}{2}} \binom{n}{k}_{p,q} x^k.
\]

Again by some simple calculations and using the property of \((p, q)\)-integers, we get \((p, q)\)-analogue of Pascal’s relation as follows:

\[
\binom{n}{k}_{p,q} = q^{n-k} \binom{n-1}{k-1}_{p,q} + p^k \binom{n-1}{k}_{p,q}, \quad (1.4)
\]

\[
\binom{n}{k}_{p,q} = p^{n-k} \binom{n-1}{k-1}_{p,q} + q^k \binom{n-1}{k}_{p,q}. \quad (1.5)
\]

We recall some results from [17] for Lupas \((p, q)\)-Bernstein operators, which reproduces linear and constant functions.
Some auxiliary results:

(1) \( L_{n}^{p,q}(1, \frac{u}{u+1}) = 1 \)

(2) \( L_{n}^{p,q}(t, \frac{u}{u+1}) = \frac{u}{u+1} \)

(3) \( L_{n}^{p,q}(t^2, \frac{u}{u+1}) = \frac{u}{u+1} \frac{u^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1} \frac{[n-1]_{p,q}}{[n]_{p,q}} \)

or equivalently for \( x = \frac{u}{u+1} \)

\[
L_{n}^{p,q}(1, x) = 1, \quad (1.6) \\
L_{n}^{p,q}(t, x) = x, \quad (1.7) \\
L_{n}^{p,q}(t^2; x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2x^2}{p(1-x) + qx} \frac{[n-1]_{p,q}}{[n]_{p,q}}. \quad (1.8)
\]

2. Construction of \((p, q)\)-Lupaș Stancu Operators

In this section, we introduce \((p, q)\)-Lupaș Stancu operators as follows:

For any \( p > 0 \) and \( q > 0 \), the linear operators \( L_{p,q}^{n} : C[0,1] \rightarrow C[0,1] \)

\[
L_{n}^{\alpha,\beta}(f; x) = \sum_{k=0}^{n} f \left( \frac{p^{n-k}}{[k]_{p,q}} \alpha + \frac{[n-1]_{p,q}}{[n]_{p,q}} \beta \right) b_{k,n}^{p,q}(t) \quad (2.1)
\]

and \( b_{k,n}^{p,q}(t) \) is given by

\[
b_{k,n}^{p,q}(t) = \frac{n!}{k!(n-k)!} \frac{p^{n-k-\alpha} q^{k-1}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}}, \quad (2.2)
\]

where \( 0 < \alpha < \beta \).

We give some equalities for operators (2.1) in the following lemma.

**Lemma 4.1.** The following equalities are true:

(i) \( L_{n}^{\alpha,\beta}(1; x) = 1 \),

(ii) \( L_{n}^{\alpha,\beta}(t; x) = \frac{[n]_{p,q}^\alpha}{[n]_{p,q}^{\alpha+\beta}}, \)

(iii) \( L_{n}^{\alpha,\beta}(t^2; x) = \frac{1}{\prod \{p^{j-1}(1-t) + q^{j-1}t\}} \left( \frac{q^2 x^2 [n-1]_{p,q}}{p(1-x) + qx} x^2 + \frac{[n]_{p,q}^{2\alpha + p^{n-1}}}{([n]_{p,q}^{\alpha+\beta})^2} x + \frac{\alpha^2}{([n]_{p,q}^{\alpha+\beta})^2} \right). \)
Proof. Proof of part (i) is obvious.

\[ L_{n,p,q}(t; x) = \sum_{k=0}^{\infty} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{p,q}^{k,n}(t) \]

\[ = \frac{[n]_{p,q}}{[n]_{p,q} + [\beta]} L_{n,p,q}^{n}(t; x) + \frac{[\alpha]}{[n]_{p,q} + [\beta]} L_{n,p,q}^{n}(1; x). \]

So from inequalities (1.6) and (1.7), we get the result.

Proof (iii)

\[ L_{n,p,q}(t^2; x) = \sum_{k=0}^{\infty} \left( \frac{p^{n-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) b_{p,q}^{k,n}(t) \]

\[ = \frac{1}{([n]_{p,q} + \beta)^2} \left[ p^{2n-2k} [k]_{p,q}^{2} b_{p,q}^{k,n}(t) \right. \]

\[ + 2\alpha p^{n-k} [k]_{p,q} b_{p,q}^{k,n}(t) + \alpha^2 b_{p,q}^{k,n}(t) \]

\[ = \frac{1}{([n]_{p,q} + \beta)^2} \left[ A + B + C \right]. \]

\[ A = p^{2n} \sum_{k=0}^{\infty} \frac{[k]^2}{p^{2k}} \frac{\binom{n}{k} p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} t^k (1-t)^{n-k}}{\prod_{j=1}^{n} \{p^{j-1}(1-t) + q^{j-1}t\}} \]

\[ A = [n]_{p,q} p^{2n} \sum_{k=1}^{\infty} \frac{[k]}{p^{2k}} \frac{\binom{n-1}{k-1} p^{(n-k)(n-k-1)/2} q^{k(k+1)/2} u^k}{\prod_{j=1}^{n} \{p^{j-1} + q^{j-1}u\}}. \]

On shifting the limits and on replacing \( k \) by \( k+1 \), we get

\[ A = [n]_{p,q} p^{2n} \sum_{k=1}^{\infty} \frac{[k+1]}{p^{2k+2}} \frac{\binom{n-1}{k} p^{(n-k)(n-k-1)/2} q^{k(k+1)/2} u^k}{\prod_{j=1}^{n} \{p^{j} + q^{j}u\}} \]

\[ = [n]_{p,q} p^{n} \frac{u}{u+1} \sum_{k=0}^{n-1} \frac{[k+1]}{p^{k+2}} \frac{\binom{n-1}{k} p^{(n-k)(n-k-1)/2} q^{k(k+1)/2} (\frac{m}{p})^k}{\prod_{j=0}^{n-2} \{p^{j} + q^{j}u\}}. \]
Using \([k + 1]_{p,q} = p^k + q[k]_{p,q}\), we get our desired result:

\[
A = [n]_{p,q} p^{n-1} \frac{u}{u + 1} \sum_{k=0}^{n-1} \frac{[p^k + q[k]]}{p^{k+2}} \left[ \frac{n - 1}{k} \right]_{p,q} p^{(n-k-1)(n-k-2)} q^{k(k+1)} \left( \frac{qu}{p} \right)^k,
\]

\[
= [n]_{p,q} p^{n-1} \frac{u}{u + 1} + \frac{q^2u^2[n]_{p,q}[n - 1]}{(u + 1)(p + qa)},
\]
equivalently

\[
A = [n]_{p,q} p^{n-1} x + \frac{q^2[n]_{p,q}[n - 1]}{(p(1 - x) + qx)} x^2.
\]

Similarly

\[
B = 2\alpha p^n \sum_{k=0}^{n} \frac{[k]}{p^k} \left[ \frac{n - 1}{k} \right]_{p,q} p^{(n-k)(n-k-1)} q^{k(k+1)} t^k (1 - t)^{n-k},
\]

\[
= 2\alpha [n]_{p,q} p^n \sum_{k=1}^{n} \frac{1}{p^k} \left[ \frac{n - 1}{k - 1} \right]_{p,q} p^{(n-k)(n-k-1)} q^{k(k+1)} u^k,
\]

After shifting the limits and on replacing \(k\) by \(k + 1\), we get

\[
B = 2\alpha [n]_{p,q} x.
\]

Similarly

\[
C = \alpha^2 \sum_{k=0}^{n} \frac{1}{p^{k+2}} \left[ \frac{n}{k} \right]_{p,q} p^{(n-k)(n-k-1)} q^{k(k+1)} t^k (1 - t)^{n-k},
\]

\[
= \alpha^2.
\]
Theorem 2.1. Let $0 < q_n < p_n \leq 1$ such that $\lim_{n \to \infty} p_n = 1$, $\lim_{n \to \infty} q_n = 1$, $\lim_{n \to \infty} q_n^n = 1$ and for $f \in C[0,1]$, we have $\lim_n |L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)| = 0$.

Proof. Let us recall the following Korovkin’s theorem see [20]. Let $(T_n)$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. Then $\lim_{n \to \infty} \|T_n(f, x) - f(x)\|_{C[a,b]} = 0$, for all $f \in C[a,b]$ if and only if $\lim_{n \to \infty} \|T_n(f_i, x) - f_i(x)\|_{C[a,b]} = 0$, for $i = 0, 1, 2$, where $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$.

3. The Rate of Convergence

In this section, we compute the rates of convergence of the operators $L_{\alpha,\beta}^{n,p,q}(f; x)$ to the functions $f$ by means of modulus of continuity, elements of Lipschitz class and peetre’s K-functional.

Let $f \in C[0,1]$. The modulus of continuity of $f$ denoted by $\omega(f, \delta)$ is defined as:

$$\omega(f, \delta) = \sup_{y, x \in [0,1], |y-x| < \delta} |f(y) - f(x)|.$$ 

where $w(f; \delta)$ satisfies the following conditions: for all $f \in C[0,1],$

$$\lim_{\delta \to 0} w(f; \delta) = 0.$$ 

and

$$|f(y) - f(x)| \leq w(f; \delta) \left( \frac{|y-x|}{\delta} + 1 \right).$$

Theorem 3.1. Let $0 < q < p \leq 1$, and $f \in C[0,1]$, and $\delta > 0$, we have

$$\|L_{\alpha,\beta}^{n,p,q}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n)$$

where

$$\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1-x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) \right]^\frac{1}{2}.$$
Proof. From lemma (4.1) we have

\[
|L_{n,p,q}^{\alpha,\beta}(t - x)^2; x) = \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1 - x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right)x^2 \\
+ \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right)x + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}.
\]

(3.3)

For \( x \in [0, 1] \), we take

\[
|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left( L_{n,p,q}^{\alpha,\beta}(t - x)^2; x \right)^{\frac{1}{2}} \right\},
\]

then we get

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq w(f; \delta) \left\{ 1 + \frac{1}{\delta} \left( \frac{1}{([n]_{p,q} + \beta)^2} \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{p(1 - x) + qx} \\
- \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) + \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \\
+ \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right\}^{\frac{1}{2}}.
\]

If we choose

\[
\delta_n = \left[ \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{([n]_{p,q} + \beta)^2(p(1 - x) + qx)} - \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right) \\
+ \left( \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q} + \beta)^2} \right) + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right]^{\frac{1}{2}}.
\]

Then we have

\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2\omega(f; \delta_n).
\]

So we have the desired result. \( \square \)

Now we compute the approximation order of operator \( L_{n,p,q}^{\alpha,\beta} \) in term of the elements of the usual Lipschitz class.
Let $f \in C[0,1]$ and $0 < \rho \leq 1$. We recall that $f$ belongs to $\text{Lip}_M(\rho)$ if the inequality

$$|f(x) - f(y)| \leq M|x - y|^{\rho}; \text{ for all } x, y \in [0,1]$$

(3.4)

holds.

**Theorem 3.2.** For all $f \in \text{Lip}_M(\rho)$

$$\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq M\delta_n^\rho$$

where

$$\delta_n = \left[ \left( \frac{q^2 [n]_{p,q} [n-1]_{p,q}}{([n]_{p,q} + \beta)^2} (p(1-x) + qx) \right) \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} \right]^{\frac{1}{2}}$$

and $M$ is a positive constant.

**Proof.** Let $f \in \text{Lip}_M(\rho)$ and $0 < \rho \leq 1$. by (3.4) and linearity and monotonicity of $L_{n,p,q}^{\alpha,\beta}$ then we have

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq L_{n,p,q}^{\alpha,\beta}(|f(t) - f(x)|; x)$$

$$\leq L_{n,p,q}^{\alpha,\beta}(|t - x|^\rho; x).$$

Applying the Holder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq (L_{n,p,q}^{\alpha,\beta}(|t - x|^2; x))^\frac{1}{2}.$$

(3.5)

if we choose $\delta = \delta_n$ as above, then proof is completed.

Finally, we will study the rate of convergence of the positive linear operators $L_{n,p,q}^{\alpha,\beta}$ by means of the Peetre’s K-functionals.

$C^2[0,1]$ : The space of those functions $f$ for which $f, f', f'' \in C[0,1]$. we recall the following norm in the space $C^2[0,1]$:

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$

We consider the following Peetre’s K-functional

$$K(f, \delta) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\|_{C[0,1]} + \delta \|g\|_{C[0,1]} \right\}.$$
Theorem 3.3. Let \( f \in C[0, 1] \). Then we have
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f; \delta_n)
\]
Where \( K(f; \delta_n) \) is Peetre’s functional and
\[
\delta_n = \frac{1}{2} \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{[n]_{p,q} + \beta} \left( p(1 - x) + qx \right) \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} + \frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \left[ \alpha + [\beta] \right].
\]

Proof. Let \( g \in C^2[0,1] \). If we use the Taylor’s expansion of the function \( g \) at \( s = x \), we have
\[
g(s) = g(x) + (s - x)g'(x) + \frac{(s - x)^2}{2} g''(x).
\]
Hence we get
\[
\|L_{n,p,q}^{\alpha,\beta}(f; x) - f(x)\|_{C[0,1]} \leq \|L_{n,p,q}^{\alpha,\beta}((s - x); x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}
+ \frac{1}{2} \|L_{n,p,q}^{\alpha,\beta}((s - x)^2; x)\|_{C[0,1]}\|g(x)\|_{C^2[0,1]}.
\]

(3.6)

From the lemma (2.1) we have
\[
\|L_{n,p,q}^{\alpha,\beta}((s - x); x)\|_{C[0,1]} \leq \frac{[\alpha] + [\beta]}{[n]_{p,q} + [\beta]}.
\]

(3.7)

So if we use (3.3) and (3.7) in (3.6), then we get
\[
\|L_{n,p,q}^{\alpha,\beta}(g; x) - g(x)\|_{C[0,1]} \leq \frac{1}{2} \left( \frac{q^2[n]_{p,q}[n - 1]_{p,q}}{[n]_{p,q} + \beta} \left( p(1 - x) + qx \right) \frac{[n]_{p,q} - \beta}{[n]_{p,q} + \beta} + \frac{1}{4} \frac{p^{n-1}[n]_{p,q} - 2\alpha \beta}{([n]_{p,q} + \beta)^2} \right) + \frac{1}{4} \frac{\alpha^2}{([n]_{p,q} + \beta)^2} + \frac{1}{2} \left[ \alpha + [\beta] \right]\|g(x)\|_{C[0,1]}.
\]

(3.8)

(3.9)

On the other hand, we can write
\[ |L_{n,p,q}^{α,β}(f; x) - f(x)| \leq |L_{n,p,q}^{α,β}(f - g; x)| + |L_{n,p,q}^{α,β}(g; x) - g(x)| + |f(x) - g(x)|. \]

If we take the maximum on \([0, 1]\), we have
\[ \|L_{n,p,q}^{α,β}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \|L_{n,p,q}^{α,β}(g; x) - g(x)\|_{C[0,1]}. \quad (3.10) \]

If we consider (3.8) in (3.10), we obtain
\[
\|L_{n,p,q}^{α,β}(f; x) - f(x)\|_{C[0,1]} \leq 2\|f - g\|_{C[0,1]} + \left[ \frac{1}{4} \left( \frac{q[n_{p,q}][n - 1]_{p,q}}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right) + \frac{1}{4} \left( \frac{p^{n-1}[n_{p,q}] - 2αβ}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right) \\
+ \frac{1}{4} \left( \frac{α^2}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right) + \frac{1}{2} \left[ \frac{α}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right] \right] \|g(x)\|_{C^2[0,1]}. 
\]

If we choose
\[
δ_n = \frac{1}{4} \left( \frac{q^2[n_{p,q}][n - 1]_{p,q}}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} - \frac{[n_{p,q}] - \frac{1}{\beta}}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right) + \frac{1}{4} \left( \frac{p^{n-1}[n_{p,q}] - 2αβ}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right) + \frac{1}{4} \left( \frac{α^2}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right) + \frac{1}{2} \left[ \frac{α}{\frac{1}{n_{p,q}} + \frac{1}{\beta}} \right],
\]
then we get
\[
\|L_{n,p,q}^{α,β}(f; x) - f(x)\|_{C[0,1]} \leq 2 \left\{ \|f - g\|_{C[0,1]} + δ_n\|g(x)\|_{C^2[0,1]} \right\}. \quad (3.11)
\]

Finally, one can observe that if we take the infimum of both sides of above inequality for the function \(g \in C^2[0,1]\), we can find
\[
\|L_{n,p,q}^{α,β}(f; x) - f(x)\|_{C[0,1]} \leq 2K(f, δ_n).
\]

\[\square\]

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence. The statistical convergence which was introduced by Fast [41] in 1951, is an important research area in approximation theory. In [41], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.
Recently, statistical approximation properties of many operators are investigated in [4, 25, 26, 29, 30].

A sequence $x = (x_k)$ is said to be statistically convergent to a number $L$ if for every $\varepsilon > 0$,

$$\delta \{ K \in \mathbb{N} : |x_k - L| \geq \varepsilon \} = 0,$$

where $\delta(K)$ is the natural density of the set $K \subseteq \mathbb{N}$.

The density of subset $K \subseteq \mathbb{N}$ is defined by

$$\delta(K) := \lim_{n \to \infty} \frac{1}{n} \{ \text{the number } k \leq n : k \in K \}$$

whenever the limit exists.

For instance, $\delta(\mathbb{N}) = 1$, $\delta\{2K : k \in \mathbb{N}\} = \frac{1}{2}$ and $\delta\{k^2 : K \in \mathbb{N}\} = 0$.

To emphasize the importance of the statistical convergence, we have an example: The sequence

$$X_k = \begin{cases} L_1; & \text{if } k = m^2, \\ L_2; & \text{if } k \neq m^2. \end{cases}$$

where $m \in \mathbb{N}$ (4.1)

is statistically convergent to $L_2$ but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statistically convergent but not conversely.

Now we consider sequences $q = q_n$ and $p = p_n$ such that:

$$st - \lim_n q_n = 1, \quad st - \lim_n p_n = 1, \quad \text{and } st - \lim_n q_n^a = 1.$$ (4.2)

Gadjiev and Orhan [41] gave the following theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.
Theorem 4.1. If $A_n$ be the sequence of linear positive operators from $C[a, b]$ to $C[a, b]$ satisfies the conditions

$$\text{st} - \lim_{n} \|A_n((t^\nu; x)) - (x)^\nu\|_{C}[0, 1] = 0$$

for $\nu = 0, 1, 2$.

then for any function $f \in C[a, b]$,

$$\text{st} - \lim_{n} \|A_n(f; .) - f\|_{C}[a, b] = 0.$$ 

Now we will discuss the rates of statistical convergence of $L_{n,p,q}^{\alpha,\beta}$ operators.

Remark 4.2. For $q \in (0, 1)$ and $p \in (q, 1]$ it is obvious that

$$\lim_{n \to \infty} [n]_{p,q} = \begin{cases} 0, & \text{when } p, q \in (0, 1) \\ \frac{1}{1 - q}, & \text{when } p = 1 \text{ and } q \in (0, 1). \end{cases}$$

In order to reach to convergence results of the operator $L_{n,p,q}^{\alpha,\beta}(f; x)$, we take a sequence $q_n \in (0, 1)$ and $p_n \in (q, 1]$ such that

$$\lim_{n \to \infty} p_n = 1, \quad \lim_{n \to \infty} q_n = 1.$$ 

So we get

$$\lim_{n \to \infty} [n]_{p_n, q_n} = \infty.$$ 

Theorem 4.3. Let $L_{n,p,q}^{\alpha,\beta}$ be the sequence of operators and the sequences $p = p_n$ and $q = q_n$ satisfies Remark 4.2 then for any function $f \in C[0, 1]$

$$\text{st} - \lim_{n} \|L_{n,p_n,q_n}^{\alpha,\beta}(f; .) - f\| = 0. \quad (4.3)$$

Proof. Clearly for $\nu = 0$,

$$L_{n,p_n,q_n}^{\alpha,\beta}(1, x) = 1,$$

which implies

$$\text{st} - \lim_{n} \|L_{n,p_n,q_n}^{\alpha,\beta}(1; x) - 1\| = 0.$$ 

For $\nu = 1$

$$\|L_{n,p_n,q_n}^{\alpha,\beta}(t; x) - x\| \leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} x + \frac{\alpha}{[n]_{p_n,q_n} + \beta} - x \right|$$

$$= \left| \left( \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1 \right) x + \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right|$$

$$\leq \left| \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n} + \beta} - 1 \right| + \left| \frac{\alpha}{[n]_{p_n,q_n} + \beta} \right|.$$
For a given $\epsilon > 0$, let us define the following sets.

\[
U = \{ n : \| L_{n,p,q}^{\alpha,\beta}(t;x) - x \| \geq \epsilon \}
\]
\[
U' = \{ n : 1 - \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} \geq \epsilon \}
\]
\[
U'' = \{ n : \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} \geq \epsilon \}.
\]

It is obvious that $U \subseteq U'' \cup U'$.

So using

\[
\delta \{ k \leq n : 1 - \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} \geq \epsilon \},
\]
then we get

\[
st - \lim_n \| L_{n,p,q}^{\alpha,\beta}(t;x) - x \| = 0. \tag{4.4}
\]

Lastly for $\nu = 2$, we have

\[
\| L_{n,p,q}^{\alpha,\beta}(t^2 : x) - x^2 \| \leq \left| \frac{q^2[n]_{p,q} [n-1]_{p,q} - 1}{p(1-x) + qx} \right| \left( \frac{[n]_{p,q} + \beta}{\beta} \right)^2
\]
\[
+ \left| \frac{[n]_{p,q} (2\alpha + p^{n-1})}{\beta} \right| x + \left| \frac{\alpha^2}{\beta} \right|.
\]

If we choose

\[
\alpha_n = \frac{q^2[n]_{p,q} [n-1]_{p,q} - 1}{p(1-x) + qx} \left( \frac{[n]_{p,q} + \beta}{\beta} \right)^2 - 1
\]
\[
\beta_n = \frac{[n]_{p,q} (2\alpha + p^{n-1})}{\beta}
\]
\[
\gamma_n = \frac{\alpha^2}{\beta}.
\]

Then

\[
st - \lim_n \alpha_n = st - \lim_n \beta_n = st - \lim_n \gamma_n = 0.
\]

Now given $\epsilon > 0$, we define the following four sets:

\[
U = \| L_{n,p,q}^{\alpha,\beta}(t^2 : x) - x^2 \| \geq \epsilon,
\]
\[
U_1 = \{ n : \alpha_n \geq \frac{\epsilon}{3} \},
\]
\[
U_2 = \{ n : \beta_n \geq \frac{\epsilon}{3} \},
\]
\[ U_3 = \{ n : \gamma_n \geq \epsilon_3 \} . \]

It is obvious that \( U \subseteq U_1 \cup U_2 \cup U_3 \). Thus we obtain

\[
\delta\{ K \leq n : \| L_{\alpha,\beta}^{n,p,q}(t^2 : x) - x^2 \| \geq \epsilon \} \\
\leq \delta\{ K \leq n : \alpha_n \geq \frac{\epsilon}{3} \} + \delta\{ K \leq n : \beta_n \geq \frac{\epsilon}{3} \} + \delta\{ K \leq n : \gamma_n \geq \frac{\epsilon}{3} \}.
\]

So the right hand side of the inequalities is zero.

Then

\[
st - \lim n \| L_{\alpha,\beta}^{n,p,q}(t;x) - x \| = 0
\]

holds and thus the proof is completed.

\[ \square \]

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