Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 2 (2019), pp 173-184
DOI: 10.7508/ijmsi.2019.02.015

Chromatic Harmonic Indices and Chromatic Harmonic Polynomials of Certain Graphs

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Abstract. In the main this paper introduces the concept of chromatic harmonic polynomials denoted, $H^\chi(G,x)$ and chromatic harmonic indices denoted, $H^\chi(G)$ of a graph G. The new concept is then applied to finding explicit formula for the minimum (maximum) chromatic harmonic polynomials and the minimum (maximum) chromatic harmonic index of certain graphs. It is also applied to split graphs and certain derivative split graphs.

Keywords: Chromatic harmonic index, Chromatic harmonic polynomial, Split graph, Derivative split graph.

2000 Mathematics Subject Classification: 05C15, 05C38, 05C75, 05C85.

1. Introduction

For general notation and concepts in graphs and digraphs see [1] [7]. Unless mentioned otherwise all graphs are simple, connected and undirected graphs. In this article a graph G will have order $n \geq 2$ with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and size $p \geq 1$ with edge set $E(G) = \{e_1, e_2, e_3, \dots, e_p\}$,

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Received 15 January 2017; Accepted 23 July 2018

denoted as $\nu(G) = n$ and $\varepsilon(G) = p$. An edge $e_i = v_iv_j$ means that the vertices v_i, v_j are adjacent. A multivariate polynomial over a field whose Laplacian is zero is termed as Harmonic polynomial. They form a vector subspace of the vector space of polynomials over the field.

In [8] Zhong introduced the harmonic index for graphs. Harmonic index is one of the most important indices in chemical and mathematical fields. It is a variant of the Randic index which is the most successful molecular descriptor in structure-property and structure activity relationship studies. Very recently in [2], Iranmanesh et. al introduced the concept of the harmonic polynomial of a graph G as

Definition 1.1. [2] $H(G,x) = \sum\limits_{uv \in E(G)} 2x^{d_G(u)+d_G(v)-1}$, where $\int_0^1 H(G,x) = H(G)$.

Researchers are interested in considering the relationship between the harmonic index and the eigenvalues of graphs, determining the minimum and maximum values of the harmonic index and, estimating the bounds for $H(G)$.
In [8] the authors established explicit formulas for the harmonic polynomial of several classes of graphs.

It is observed that most structural indices of kind, are defined in terms of the vertex degree in G. The variation we will consider is that of the colour of a vertex when applying what is known to be a minimum parameter chromatic colouring to G [4].

2. Chromatic Harmonic Polynomial and Chromatic Harmonic Index

One may recall that if $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$ is a set of distinct colours, a proper vertex colouring of a graph G denoted $\varphi : V(G) \mapsto \mathcal{C}$ is a vertex colouring such that no two distinct adjacent vertices have the same colour. The cardinality of a minimum set of colours which is a proper vertex colouring of G is called the chromatic number of G and is denoted $\chi(G)$. When a vertex colouring is considered with colours of minimum subscripts the colouring is called a *minimum parameter* colouring. Unless stated otherwise we consider minimum parameter colour sets throughout this paper. The number of times a colour c_i is allocated to vertices of a graph G is denoted by $\theta(c_i)$ and $\varphi : v_i \mapsto c_j$ is abbreviated, $c(v_i) = c_j$. Furthermore, we define an important derivative index that is, if $c(v_i) = c_j$ then $\iota(v_i) = j$.

Rainbow Neighborhood Convention:[5] Unless mentioned otherwise we shall consider the colours $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$ and always colour vertices with maximum c_1, followed by maximum c_2 among the remaining uncoloured vertices, ..., followed by maximum c_ℓ for the final remaining uncoloured vertices.

Note that the Rainbow Neighborhhood Convention ensures a minimum valued chromatic harmonic polynomial and therefore a minimum chromatic harmonic index. The inverse to the convention ensures the maximum valued chromatic harmonic polynomial and the maximum chromatic harmonic index. The inverse colouring requires the mapping $c_j \mapsto c_{\ell-(j-1)}$. Corresponding to the inverse colouring we define the inverse index $\iota'(v_i) = \ell - (j-1)$ if $c(v_i) = c_j$. We shall colour a graph in accordance with the Rainbow Neighborhood Convention [5]. We are now ready to introduce the definitions of the chromatic harmonic polynomials and the chromatic harmonic indices.

Definition 2.1. For a graph G and the minimum parameter colour set $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_{\chi(G)}\}$ the minimum (or maximum) chromatic harmonic polynomial (CHP^- or CHP^+) and the minimum (or maximum) chromatic harmonic index (CHI^- or CHI^+) are defined as

$$H^{\chi^-}(G,x) = \sum_{v_iv_j \in E(G)} 2x^{\iota(v_i)+\iota(v_j)}, \text{ and } H^{\chi^-}(G) = \int_0^1 H^{\chi^-}(G,x)$$

and,

$$H^{\chi^+}(G,x) = \sum_{v_iv_j \in E(G)} 2x^{\iota'(v_i)+\iota'(v_j)}, \text{ and } H^{\chi^+}(G) = \int_0^1 H^{\chi^+}(G,x)$$

Proposition 2.2. *For a complete graph K_n, $n \geq 2$,*
(1) If n is even, then

$$H^{\chi^-}(K_n,x) = H^{\chi^+}(K_n,x) = 2\cdot[x^{2n-1} + x^{2n-2} + 2(x^{2n-3} + x^{2n-4}) + 3(x^{2n-5} + x^{2n-6}) + \cdots + \tfrac{n}{2}(x^{n+2} + x^{n+1}) + (\tfrac{n}{2}-1)(x^n + x^{n-1}) + (\tfrac{n}{2}-2)(x^{n-2} + x^{n-3}) + \cdots + 2(x^6 + x^5) + x^4 + x^3],$$

(2) If n is odd, then

$$H^{\chi^-}(K_n,x) = H^{\chi^+}(K_n,x) = 2\cdot[x^{2n-1} + x^{2n-2} + 2(x^{2n-3} + x^{2n-4}) + 3(x^{2n-5} + x^{2n-6}) + \cdots + \lfloor\tfrac{n}{2}\rfloor(x^{n+3} + x^{n+2} + x^{n+1}) + (\lfloor\tfrac{n}{2}\rfloor-1)(x^n + x^{n-1}) + (\lfloor\tfrac{n}{2}\rfloor-2)(x^{n-2} + x^{n-3}) + \cdots + 2(x^6 + x^5) + x^4 + x^3].$$

Proof. For a complete graph K_n, $n \geq 2$ we have that $\theta(c_i) = 1$, $\forall c_i \in \{c_1, c_2, c_3, \dots, c_n\}$. It is known that for the integers $a < b$ there exist exactly $t = (b-a)-1$ integers which all hence, anyone say x, satisfies $a < x < b$. It implies that there are $\lfloor\frac{t}{2}\rfloor$ pairs of such inbetween integers with sum equal to $a+b$. Also, for t even we have that $\lfloor\frac{t}{2}\rfloor = \lfloor\frac{t+1}{2}\rfloor$. Clearly as a result of completeness the principle of symmetry in summation applies and both the results follow from Definition 2.1 and through immediate induction. □

Proposition 2.3. *For a cycle C_n, $n \geq 3$*
(1) When n is even,
$H^{\chi^-}(C_n, x) = H^{\chi^+}(C_n, x) = 2nx^3$, *and* $H^{\chi^-}(C_n) = H^{\chi^+}(C_n) = \frac{n}{2}$,
(2)When n is odd,
$H^{\chi^-}(C_n, x) = 2(n-2)x^3 + 2x^4 + 2x^5$, $H^{\chi^+}(C_n, x) = 2(n-2)x^5 + 2x^4 + 2x^3$, *and*
$H^{\chi^-}(C_n) = \frac{n}{2} - \frac{4}{15}$, $H^{\chi^+}(C_n) = \frac{n-2}{3} + \frac{9}{10}$.

Proof. (1) For n is even, C_n is bipartite hence, the chromatic number equals 2. Further, because $|E(C_n)| = n$ the results follow easily.
(2) For odd n, the chromatic number of C_n , $\chi(C_n) = 3$. For minimum colour sums for the edges the minimum parameter colour set $\{c_1, c_2, c_3\}$, allows exactly one vertex say, v_n with colour c_3. It follows that v_n is adjacent to vertices with colours c_1, c_2 respectively. Therefore the colour sum terms $2x^4$ and $2x^5$ follow. For all the other $n-2$ edges the colour sum term $2x^3$ applies.
For maximum colour sums for the edges the colour rotation mapping $c_i \mapsto c_{\chi-(i-1)}$ applies and the result follows along the same reasoning. □

Proposition 2.4 discuss H^{χ^-} and H^{χ^+} of the certain classes of graphs such as Π_n, $K_{m,n}$, $S_n = K_{1,n-1}$, P_n, and Q_n.

Proposition 2.4.

1. *For a prism Π_n, formed by the two cycle C_n, $n \geq 3$ and n is odd,*
   $H^{\chi^-}(\Pi_n, x) = 6(n-2)x^3 + 6x^4 + 6x^5 = \frac{3n}{2} - \frac{4}{5}$ *and,*
   $H^{\chi^+}(\Pi_n, x) = 6(n-2)x^5 + 6x^4 + 6x^3 = n + \frac{7}{10}$, *and*

   For n is even,
   $H^{\chi^-}(\Pi_n, x) = H^{\chi^+}(\Pi_n, x) = 6nx^3$.

2. *For complete bipartite graph $K_{m,n}$, where $m, n \geq 2$,*

   $$H^{\chi^-}(K_{m,n}, x) = H^{\chi^+}(K_{m,n}, x) = 2mnx^3,$$

   $$H^{\chi^-}(K_{m,n}) = H^{\chi^+}(K_{m,n}) = \frac{mn}{2}.$$

3. *For $n \geq 3$, and $S_n = K_{1,n-1}$,*

   $$H^{\chi^-}(S_n, x) = H^{\chi^+}(S_n, x) = 2(n-1)x^3,$$

   $$H^{\chi^-}(S_n) = H^{\chi^+}(S_n) = \frac{n-1}{2}.$$

4. *For Path* P_n, $n \geq 3$,
$H^{\chi^-}(P_n, x) = H^{\chi^+}(P_n, x) = 2(n-1)x^3$

$$H^{\chi^-}(P_n) = H^{\chi^+}(P_n) = \tfrac{n-1}{2}.$$

5. *For* $Q_n = K_2 \times Q_{n-1}$, $n \geq 1$,
$H^{\chi^-}(Q_n, x) = H^{\chi^+}(Q_n, x) = n2^n x^3$ *and* $H^{\chi^-}(Q_n) = H^{\chi^+}(Q_n) = n2^{n-2}$.

Proof. Consider the prism formed by the two cycle C_n, $n \geq 3$ and n is odd. Label the vertices of the respective cycles as $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ such that we have the edges, v_iu_i, $1 \leq i \leq n$. Colour the vertices as $c(v_1) = c_1, c(v_2) = c_2, c(v_3) = c_1, \cdots, c(v_{n-1}) = c_2, c(v_n) = c_3$ and $c(u_n) = c_1, c(u_1) = 2, c(u_2) = c_1, \cdots, c(u_{n-1}) = c_3$. Clearly, this vertex colouring ensures minimum colour sums for all edges and the result follows. For maximum colour sums for the edges the colour rotation mapping $c_i \mapsto c_{\chi-(i-1)}$ applies and the result follows along the same reasoning. Since all graphs $K_{m,n}$, $S_n = K_{1,n-1}$, P_n, and Q_n are bipartite, the respective chromatic number equals 2. Further, because $|E(C_n)| = n, |E(\Pi_n)| = 3n, |E(K_{m,n})| = mn, |E(S_n)| = n-1, |E(P_n)| = n-1$ and $|E(Q_n)| = n2^{(n-1)}$, one may easily check that the results follows. □

Corollary 2.5. *Any graph* G *of size* $\varepsilon(G) = p$ *and* $\chi(G) = 2$, *has* $H^{\chi^-}(G, x) = H^{\chi^+}(G, x) = 2px^3$ *and* $H^{\chi^-}(G) = H^{\chi^+}(G) = \frac{p}{2}$.

Proof. Clearly each edge $e \in E(G)$ is incident with vertices coloured c_1 and c_2, respectively. Hence, the result. □

A wide variety of remarkable graphs have chromatic number equal to 2. Invoking Corollary 2.5 to some important 2-chromatic graphs are tabled below. Table 1.

Graph G	$\nu(G)$	$\varepsilon(G)$	Degree regularity	$H^{\chi^-}(G,x) = H^{\chi^+}(G,x)$	$H^{\chi^-}(G) = H^{\chi^-}(G)$
Iofinova-Ivanov	110	165	3	$330x^3$	$\frac{165}{2}$
Balaban 10-cage	70	105	3	$210x^3$	$\frac{105}{2}$
Cubicle	8	12	3	$24x^3$	6
Dyck	32	48	3	$96x^3$	24
Ellingham-Horton	54(78)	81(167)	3	$162x^3(334x^3)$	$\frac{81}{2}(\frac{167}{2})$
F_26A	26	39	3	$78x^3$	$\frac{39}{2}$
Folkman	20	40	4	$80x^3$	20
Foster	90	135	3	$270x^3$	$\frac{135}{2}$
Franklin	12	18	3	$38x^3$	9
Gray	54	81	3	$162x^3$	$\frac{81}{2}$
Harries	70	105	3	$210x^3$	$\frac{105}{2}$
Heawood	14	21	3	$42x^3$	$\frac{21}{2}$
Hoffman	16	32	4	$64x^3$	16
Horton	96	144	3	$288x^3$	72
Ljubljana	112	168	3	$236x^3$	84
Naura	24	36	3	$72x^3$	18
Pappus	18	27	3	$54x^3$	$\frac{27}{2}$
Tutte-Coxeter	30	45	3	$90x^3$	$\frac{45}{2}$

2.1. **Application in mathematical chemistry.** Figure 1 depicts the molecular structure of $TUC_4C_8[m,n]$ carbon nanotubes together with the graphical representation where vertices represent carbon atoms and edges represent bondings. Also see [3].

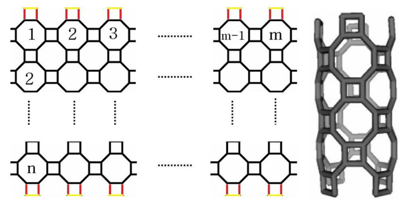

Figure 1. Molecular structure of $TUC_4C_8[m,n]$ carbon nanotubes.

Considering Figure 1 it is straightforward to verify that $TUC_4C_8[m,n]$, $m,n \in \mathbb{N}$ has $\chi(TUC_4C_8[m,n]) = 2$ and $\varepsilon(TUC_4C_8[m,n]) = 4(m+3mn)$. Therefore, $H^{\chi^-}(TUC_4C_8[m,n]) = H^{\chi^+}(TUC_4C_8[m,n]) = 8(m+3mn)x^3$. Also see [3]. Figure 2 depicts the molecular structure of $TUC_4[m,n]$ carbon nanotubes. Also see [3].

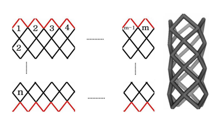

Figure 2. Molecular structure of $TUC_4[m,n]$ carbon nanotubes.

Considering Figure 2 it is straightforward to verify that the molecular graph of $TUC_4[m,n]$, $m,n \in \mathbb{N}$ nanotube has $2m(n+1)$ vertices and $2m(2n+1)$ edges. Also $\chi(TUC_4[m,n]) = 2$ therefore, $H^{\chi^-}(TUC_4[m,n]) = H^{\chi^+}(TUC_4[m,n]) = 4m(2n+1)x^3$.

Remark 2.6. For more generalised applications of vertex colouring such as locating certain technology at vertices the minimum parameter colour set could be the set $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell; \ell \geq \chi(G)\}$. It implies that different chromatic colourings in accordance with the Rainbow Neighborhood Convention are possible. Thus, for a particular chromatic colouring, a minimum (or maximum) chromatic harmonic polynomial and a minimum (or maximum) chromatic harmonic index can be derived.
Denote these general cases by $H^{\chi_c^-}(G,x), H^{\chi_c^+}(G,x)$ and
$H^{\chi_c^-}(G), H^{\chi_c^+}(G)$, respectively.

Hence, we have

Theorem 2.7.

1. *For cycle* C_n

a. *For* $n \geq 3$, *and* n *is even,*
$2nx^3 \leq H^{\chi_c^-}(C_n, x) = H^{\chi_c^+}(C_n, x) \leq 2nx^{2\ell-1}$,
$\frac{n}{2} \leq H^{\chi_c^-}(C_n) = H^{\chi_c^+}(C_n) \leq \frac{n}{\ell}$.

b. *For* $n \geq 3$, *and* n *is odd,*
$2(n-2)x^3+2x^4+2x^5 \leq H^{\chi_c^-}(C_n, x) \leq 2(n-2)x^{2\ell-3}+2x^{2\ell-2}+2x^{2\ell-1}$,
$2(n-2)x^5+2x^4+2x^3 \leq H^{\chi_c^+}(C_n, x) \leq 2(n-2)x^{2\ell-5}+2x^{2\ell-6}+2x^{2\ell-7}$,
and
$\frac{n}{2} - \frac{4}{15} \leq H^{\chi_c^-}(C_n) \leq \frac{n-2}{\ell-1} + \frac{4\ell-1}{\ell(2\ell-1)}$,
$\frac{n-2}{3} + \frac{9}{10} \leq H^{\chi_c^+}(C_n) = \frac{n-2}{\ell-2} + \frac{4\ell-11}{(2\ell-5)(\ell-3)}$.

2. *For a prism* Π_n,

a. *For* $n \geq 3$, *and* n *is even,*
$6nx^3 \leq H^{\chi_c^-}(\Pi_n, x) = H^{\chi_c^+}(\Pi_n, x) \leq 6nx^{2\ell-3}$, *and*
$\frac{3n}{2} \leq H^{\chi_c^-}(\Pi_n) = H^{\chi_c^+}(\Pi_n) \leq \frac{3n}{\ell-1}$.

b. *For* $n \geq 3$, *and* n *is odd,*
$6(n-2)x^3+6x^4+6x^5 \leq H^{\chi_c^-}(\Pi_n, x) \leq 6(n-2)x^{2\ell-3}+6x^{2\ell-2}+6x^{2\ell-1}$,
$6(n-2)x^5+6x^4+6x^3 \leq H^{\chi_c^+}(\Pi_n, x) \leq 6(n-2)x^{2\ell-5}+6x^{2\ell-6}+6x^{2\ell-7}$,
$\frac{3n}{2} - \frac{4}{5} \leq H^{\chi_c^-}(\Pi_n) \leq \frac{3}{\ell-1}(n - \frac{2\ell-1}{\ell})$,
$n + \frac{7}{10} \leq H^{\chi_c^+}(C_n) \leq \frac{3(n-2)}{\ell-2} + \frac{3(4\ell-11)}{(2\ell-5)(\ell-3)}$.

3. *For complete graph* $K_{m,n}$, $m, n \geq 2$,
$2mnx^3 \leq H^{\chi_c^-}(K_{m,n}, x) \leq H^{\chi_c^+}(K_{m,n}, x) \leq 2mnx^{2\ell-1}$,
$\frac{mn}{2} \leq H^{\chi_c^-}(K_{m,n}) = H^{\chi_c^+}(K_{m,n}) \leq \frac{mn}{\ell}$.

4. *For* $S_n = K_{1,n-1}$, $n \geq 3$,
$2(n-1)x^3 \leq H^{\chi_c^-}(S_n, x) = H^{\chi_c^+}(S_n, x) \leq 2(n-1)x^{2\ell-1}$,
$\frac{n-1}{2} \leq H^{\chi_c^-}(S_n) = H^{\chi_c^+}(S_n) \leq \frac{n-1}{\ell}$.

5. *For path* P_n, $n \geq 3$,
$2(n-1)x^3 \leq H^{\chi_c^-}(P_n, x) = H^{\chi_c^+}(P_n, x) \leq 2(n-1)x^{2\ell-1}$,
$\frac{n-1}{2} \leq H^{\chi_c^-}(P_n) = H^{\chi_c^+}(P_n) \leq \frac{n-1}{\ell}$.

6. *For* Q_n, $n \geq 1$,
$n2^nx^3 \leq H^{\chi_c^-}(Q_n, x) = H^{\chi_c^+}(Q_n, x) \leq n2^nx^{2\ell-1}$,
$n2^{n-2} \leq H^{\chi_c^-}(Q_n) = H^{\chi_c^+}(Q_n) \leq \frac{n2^{n-1}}{\ell}$.

Remark 2.8. It is important to note that in Theorem 2.7, we applied the $min\{min\}$, the $max\{min\}$, the $min\{max\}$ and the $max\{max\}$ principles. Hence for a graph G there is no relation between $max(H^{\chi_{\mathcal{C}}^{-}}(G,x))$ and $min(H^{\chi_{\mathcal{C}}^{+}}(G,x))$ or $max(H^{\chi_{\mathcal{C}}^{-}}(G))$ and $min(H^{\chi_{\mathcal{C}}^{+}}(G))$.

2.2. **Results for split graphs.** It follows that a connected graph is 1-critical in respect of its CHP and CHI in that the addition (or deletion) of a vertex (or vertices) or the addition (or deletion) of an edge (or edges) changes the outcome thereof. Numerous well-defined graph structural derivatives have been studied. For example, inserting a vertex into a single edge of certain graphs can change the chromatic number. For example, inserting a vertex into into a single edge of a cycle C_n, n is even to obtain a cycle C_{n+1}, $n+1$ is odd and vice versa. In a graph where the chromatic number remains the same, an additional polynomial term results.

We further our analysis by considering a split graph. Recall that a split graph is a graph G for which the vertex set $V(G)$ can be partitioned into two sets say V_1, V_2 such that the induced graph $\langle V_1 \rangle$ is a clique and V_2 is an independent set. Furthermore a *maximum split graph embodiment* of G has $|V_2|$ a maximum. The aforesaid means that all vertices in a clique of a split graph that are not adjacent to a vertex in the independent set V_2, must be an element of V_2. It also implies minimum clique order (or clique size). A general split graph embodiment G^s of a graph G is the graph for which the vertex set G has been partitioned into two sets V_1, V_2, and $|V_2|$ a maximum such that V_2 is an independent set. Any connected bipartite graph $B_{m,n}$ is a general split graph embodiment.

Theorem 2.9. *For a maximum split graph embodiment of G of order $n \geq 2$ and clique K_t and $\mathcal{C} = \{c_2, c_3, c_4, \ldots, c_{t+1}\}$, $\mathcal{C}' = \{c_1, c_2, c_3, \ldots, c_t\}$, we have that:*

(1) $$H^{\chi^-}(G,x) = H^{\chi_{\mathcal{C}}^{-}}(K_t,x) + \sum_{v_iv_j \in E(G)\ and\ v_i \in V_1, v_j \in V_2} 2x^{\iota(v_i)+1}$$ *and,*

(2) $$H^{\chi^+}(G,x) = H^{\chi_{\mathcal{C}'}^{+}}(K_t,x) + \sum_{v_iv_j \in E(G)\ and\ v_i \in V_1, v_j \in V_2} 2x^{\iota'(v_i)+t+1}.$$

Proof. The proof follows from Proposition 2.2 and from the fact that $\iota'(v_i) = (t+1) - (j-1)$ if $c(v_i) = c_j$ and the observation that in K_t all colour sum terms increase by exactly 2. Also in (1) all $v_j \in V_2$ are coloured c_1. In (2) all $v_j \in V_2$ are coloured c_{t+1}. □

2.3. **Derivative split graphs.** We derive a derivative split graph from a graph G by defining the insertion of vertices into some edges of G. Note that the inserted vertices forms an independent set. Therefore a derivative split graph results in a general split graph embodiment.

Construct the derivative split graph denoted, $G^{\bullet}$ in respect of G of order n and $v_i \in V(G)$ by inserting a vertex $u_i \in U$ into edges $e_i \in E(G)$, $1 \leq i \leq \varepsilon(G)$. Since $\varepsilon(G) \geq n-1$ we will consider two cases. By convention, if $\varepsilon(G) = n-1$ we will write that $G^s = K_{\varepsilon(G),n}$ and if $\varepsilon(G) > n-1$ we will write $G^s = K_{n,\varepsilon(G)}$.

Theorem 2.10. *For a graph G, of order $n \geq 2$ we have that:*

(1) If $\varepsilon(G) = n-1$ then $H^{\chi^-}(G^{\bullet}, x) = H^{\chi^+}(G^{\bullet}, x) = 2(2n-1)x^3$,

(2) If $\varepsilon(G) > n-1$ then $H^{\chi^-}(G^{\bullet}, x) = H^{\chi^+}(G^{\bullet}, x) = 2\varepsilon(G)x^3$.

Proof. (1) If $\varepsilon(G) = n-1$ then G is a path P_n or a star S_{n-1} and in both cases, $G^{\bullet} = P_{2n-1}$. Hence, the result follows from Theorem 2.7

(2) If $\varepsilon(G) > n-1$ then $G^{\bullet}$ is a path $P_{n+\varepsilon(G)}$. Hence, the result follows from Proposition Theorem 2.7. □

Construct the derivative split graph denoted, $G_1 +^{\bullet} G_2$ in respect of $G_1 + G_2$, G_1 of order n_1 and G_2 of order n_2 and $v_i \in V(G_1)$, $w_i \in V(G_2)$ by inserting a vertex $u_i \in U$ into edges $v_i w_j$, $1 \leq i \leq \varepsilon(G_1)$, $1 \leq j \leq \varepsilon(G_2)$.

Theorem 2.11. *For graph G_1 of order n_1, $\chi(G_1) = t_1$ and graph G_2 of order n_2, $\chi(G_2) = t_2$ and $t_1 \geq t_2$ and $\mathcal{C}_1 = \{c_2, c_3, c_4, \ldots, c_{t_1+1}\}$, $\mathcal{C}_1' = \{c_1, c_2, c_3, \ldots, c_{t_1}\}$, and $\mathcal{C}_2 = \{c_2, c_3, c_4, \ldots, c_{t_2+1}\}$, $\mathcal{C}_2' = \{c_1, c_2, c_3, \ldots, c_{t_2}\}$, we have that:*

(1) $$H^{\chi^-}(G_1 +^{\bullet} G_2, x) = H^{\chi^-_{\mathcal{C}_1}}(G_1, x) + H^{\chi^-_{\mathcal{C}_2}}(G_2, x) + \sum_{v_i u_j \in E(G_1 +^{\bullet} G_2), v_i \in V(G_1)} 2x^{\iota(v_i)+1} + \sum_{w_i u_j \in E(G_1 +^{\bullet} G_2), w_i \in V(G_2)} 2x^{\iota(w_i)+1},$$

(2) $$H^{\chi^+}(G_1 +^{\bullet} G_2, x) = H^{\chi^+_{\mathcal{C}_1'}}(G_1, x) + H^{\chi^+_{\mathcal{C}_2'}}(G_2, x) + \sum_{v_i u_j \in E(G_1 +^{\bullet} G_2), v_i \in V(G_1)} 2x^{\iota'(v_i)+t_1+1} + \sum_{w_i u_j \in E(G_1 +^{\bullet} G_2), w_i \in V(G_2)} 2x^{\iota'(w_i)+t_1+1}.$$

Proof. (1). Note that $d(u_i) = 2$, $\forall i$ in such a way that each vertex u_i is adjacent to one vertex $v_j \in V(G_1)$ and to one vertex $w_k \in V(G_2)$. Denote these edges $E(U)$. Hence, $E(U)$ can be partitioned into two edge sets $E_1(U)$, $E_2(U)$ of equal cardinality, $n_1 \cdot n_2$. Without loss of generality assume $E_1(U)$ has the edges incident with vertices in $V(G_1)$ and $E_2(U)$ has the edges incident with vertices in $V(G_2)$. Furthermore U is the maximum independent set in $G_1 +^{\bullet} G_2$. Therefore to ensure minimum colour sums all $u_i \in U$ have colour c_1. It implies that the last two summation terms follow from Definition 2.1.

Furthermore, since no edge exists between a vertex $v_i \in V(G_1)$ and $w_j \in V(G_2)$ and $t_1 \geq t_2$ the colour set $\mathcal{C} = \{c_2, c_3, c_4, \ldots, c_{t_1+1}\}$ will allow a chromatic colouring of both G_1, G_2 in accordance with the Rainbow Neighborhood Convention. The aforesaid together with Definition 2.1 imply the first two terms. Hence, the result.
(2). Similar reasoning as in (1) provides the result. □

Note that each $v_i \in V(G_1)$ is adjacent to exactly n_2 vertices in U and each $w_j \in V(G_2)$ is adjacent to exactly n_1 vertices in U. Theorem 2.7 has an immediate consequence for the corona graph, $G_1 \circ G_2$ with similar vertex insertion.

Corollary 2.12. *For graph G_1 of order n_1, $\chi(G_1) = t_1$ and graph G_2 of order n_2, $\chi(G_2) = t_2$ and $t_1 \geq t_2$ and $\mathcal{C}_1 = \{c_2, c_3, c_4, \ldots, c_{t_1+1}\}$, $\mathcal{C}_1' = \{c_1, c_2, c_3, \ldots, c_{t_1}\}$, and $\mathcal{C}_2 = \{c_2, c_3, c_4, \ldots, c_{t_2+1}\}$, $\mathcal{C}_2' = \{c_1, c_2, c_3, \ldots, c_{t_2}\}$, we have that:*

(1) $H^{\chi^-}(G_1 \circ^{\bullet} G_2, x) = H^{\chi^-_{\mathcal{C}_1}}(G_1, x) + n_1 \cdot H^{\chi^-_{\mathcal{C}_2}}(G_2, x) + \sum\limits_{v_i u_j \in E(G_1 \circ^{\bullet} G_2), v_i \in V(G_1)} 2x^{\iota(v_i)+1} + \sum\limits_{w_i u_j \in E(G_1 \circ^{\bullet} G_2), w_i \in V(G_2)} 2n_1 x^{\iota(w_i)+1}$,

(2) $H^{\chi^+}(G_1 \circ^{\bullet} G_2, x) = H^{\chi^+_{\mathcal{C}_1'}}(G_1, x) + n_1 \cdot H^{\chi^+_{\mathcal{C}_2'}}(G_2, x) + \sum\limits_{v_i u_j \in E(G_1 \circ^{\bullet} G_2), v_i \in V(G_1)} 2x^{\iota'(v_i)+t_1+1} + \sum\limits_{w_i u_j \in E(G_1 \circ^{\bullet} G_2), w_i \in V(G_2)} 2n_1 x^{\iota'(w_i)+t_1+1}$.

Proof. Since for each vertex $v_i \in V(G_1)$ there exists an induced subgraph $v_i + G_2$ we have $v_i +^{\bullet} G_2$ after the defined vertex insertion. Hence, independent from G_1, n_1 such induced subgraphs exist in $G_1 \circ^{\bullet} G_2$. Invoking Theorem 2.11 the result follows. □

Corollary 2.12 gives way to a new concept called the *cluster corona* of graphs G_1, G_2. In the corona $G_1 \circ G_2$ as we know it we say, G_2 has been *corona'ed* to G_1.

Definition 2.13. For the graph G_1 of order n_1 and $k \geq 1$, $k \in \mathbb{N}$ take $n_1 k$ copies of G_2. The cluster corona denoted, $G_1(\circ^k)G_2$ is the graph obtained by corona'ing k copies of G_2 to each vertex $v_i \in V(G_1)$.

Our next result follows directly from Corollary 2.12

Corollary 2.14. *For graph G_1 of order n_1, $\chi(G_1) = t_1$ and for $k \geq 1$, $k \in \mathbb{N}$ copies of graph G_2 of order n_2, $\chi(G_2) = t_2$ and $t_1 \geq t_2$ and $\mathcal{C}_1 = \{c_2, c_3, c_4, \ldots, c_{t_1+1}\}$,*
$\mathcal{C}_1' = \{c_1, c_2, c_3, \ldots, c_{t_1}\}$, and $\mathcal{C}_2 = \{c_2, c_3, c_4, \ldots, c_{t_2+1}\}$,
$\mathcal{C}_2' = \{c_1, c_2, c_3, \ldots, c_{t_2}\}$, we have that:

(1) $H^{\chi^-}(G_1(\circ^k)^\bullet G_2, x) = H^{\chi^-_{\mathcal{C}_1}}(G_1, x) + kn_1 \cdot H^{\chi^-_{\mathcal{C}_2}}(G_2, x) + \sum\limits_{v_iu_j \in E(G_1(\circ^k)^\bullet G_2), v_i \in V(G_1)} 2kx^{\iota(v_i)+1} + \sum\limits_{w_iu_j \in E(G_1(\circ^k)^\bullet G_2), w_i \in V(G_2)} 2kn_1x^{\iota(w_i)+1}$,

(2) $H^{\chi^+}(G_1(\circ^k)^\bullet G_2, x) = H^{\chi^+_{\mathcal{C}_1'}}(G_1, x) + kn_1 \cdot H^{\chi^+_{\mathcal{C}_2'}}(G_2, x) + \sum\limits_{v_iu_j \in E(G_1(\circ^k)^\bullet G_2), v_i \in V(G_1)} 2kx^{\iota'(v_i)+t_1+1} + \sum\limits_{w_iu_j \in E(G_1(\circ^k)^\bullet G_2), w_i \in V(G_2)} 2kn_1x^{\iota'(w_i)+t_1+1}$.

Perhaps the applied value of the cluster corona lies in finding various edge-defined indices and other edge-defined invariants for the recursive corona which was introduced by Vernold Vivin and Kaliraj in [6]. The recursive corona is defined as $G_1 \circ^l G_2 = (G_1 \circ^{l-1} G_2) \circ G_2$, $l \geq 1$. Now clearly after finite number of iterations say k, there exists an *core* subgraph $G_1(\circ^k)G_2$. Thereafter a *layer* of bridges (sets of cut edges) follows to be enumerated in the edge-defined index or invariant. Following on that a well-defined number say, c of cluster corona graphs $G_2(\circ^c)G_2$ follow, and so on. We shall not report on the recursive method in further detail in this paper. Describing algorithms and analysing complexity remain open.

3. Conclusion

It is clear that a wide field of further applications are available from for example, just the small graphs. The aim of this paper is indeed to only serve as an introduction to the concept of chromatic harmonic polynomials and chromatic harmonic indices. It is almost certain that this new concept will find applications in other research streams of graph theory and mathematical chemistry.

Acknowledgments

The authors express their sincere gratitude to the referee for a very careful reading of the paper and for all the comments and valuable suggestions to make this a worthy paper.

References

1. F.Harary, *Graph Theory*, Addison-Wesley, Reading MA, 1969.
2. M. A. Iranmanesh, M. Saheli, On the harmonic index and harmonic polynomial of catepillars with diameter four, *Iranian Journal of Mathematical Chemistry*, **5**(2), (2014) 35-43.
3. M.K. Jamil, J. Kok, The Harmonic index and Harmonic polynomial of Some Carbon Nanotubes, Communicated.
4. J. Kok, N.K. Sudev, K.P. Chithra, General colouring sums of graphs, *Cogent Mathematics*, **3**, (2016), 1140002.
5. J. Kok, N.K. Sudev, M.K. Jamil, Rainbow Neighborhoods of Graphs, communicated.
6. J. Vernold, K. Kaliraj, On equitable coloring of corona of wheels, *Electronic Journal of Graph Theory and Applications*, **4**(2), (2016), 206-222.
7. B. West, *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, (1996).
8. L. Zhong, The harmonic index for graphs, *Applied Mathematics Letters*, **25**, (2012), 561-566.